Coherent confluence in modal n-Kleene algebras

International Workshop on Confluence

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Joint work with:
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30th of June 2020

Introduction

Relation algebras

- Consider the full relation algebra $\mathcal{R}(X)$ on a set X:

$$(\mathcal{P}(X \times X),\ \cup,\ \emptyset,\ \cdot\ ,\ \Delta),$$

where $\cdot$ is the relational composition and Δ is the diagonal of X.

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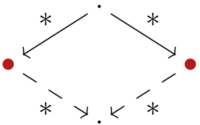

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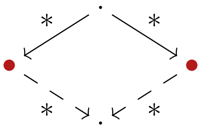

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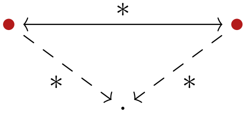

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Church-Rosser

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- A semiring $(K, +, 0, \cdot, 1)$ is a Kleene algebra if
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  - it is equipped with the Kleene star, *i.e.* a map $(-)^* : K \to K$ s.t.

$$1 + xx^* \leq x^*, \qquad 1 + x^*x \leq x^*, \qquad \text{(unfold)}$$
$$z + xy \leq y \Rightarrow x^*z \leq y, \qquad z + yx \leq y \Rightarrow zx^* \leq y \qquad \text{(induction)}$$

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Theorem (Church-Rosser à la Kleene)

Let K be a Kleene algebra and let $x, y \in K$. The following properties are equivalent:

- $x^*y^* \leq y^*x^*$
- $(x + y)^* \leq y^*x^*$

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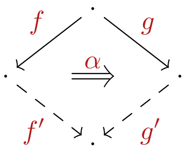

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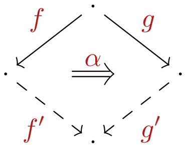

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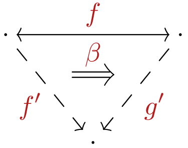

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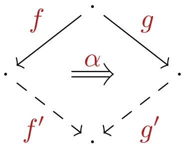

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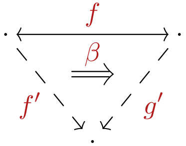

Theorem (Coherent Church-Rosser)

If P is Γ-confluent, then P is Γ-Church-Rosser.

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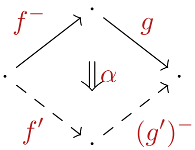

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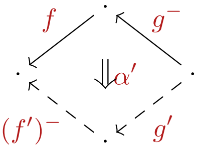

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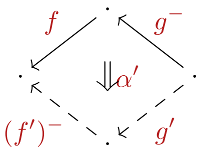

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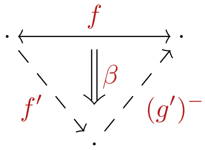

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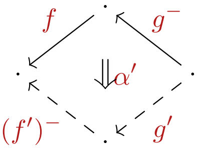

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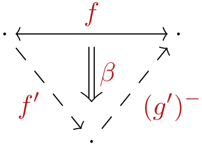

Theorem (Coherent Church-Rosser)

If Γ is a confluence filler for P, then Γ is a Church-Rosser filler for P.

Modal n-Kleene algebras

n-Semirings

- An idempotent n-semiring is a structure

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  - Interchange inequalities hold:

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    for $0 \leq i < j < n$.
  - Units are complete with respect to lower multiplications:

$$1_j \odot_i 1_j = 1_j$$

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Domain operations

- A domain n-semiring is an idempotent n-semiring $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$ equipped with domain maps

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- We denote by $S_i := d_i(S)$ the i-dimensional domain algebra of S.
- A codomain semiring is a n-semiring S equipped with maps $(r_i : S \to S)_{0 \leq i < n}$ such that $(S^{op}, (r_i)_{0 \leq i < n})$ is a domain semiring.

n-Kleene algebras

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  - $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a Kleene algebra for $0 \leq i < n$, *i.e.*

$$1_i + A \odot_i A^{*_i} \leq A^{*_i}, \qquad 1_i + A^{*_i} \odot_i A \leq A^{*_i}, \qquad \text{(unfold)}$$
$$C + A \odot_i B \leq B \quad \Rightarrow \quad A^{*_i} \odot_i C \leq B, \qquad \text{(left induction)}$$
$$C + B \odot_i A \leq B \quad \Rightarrow \quad C \odot_i A^{*_i} \leq B. \qquad \text{(right induction)}$$

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\begin{aligned}
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  - These maps must additionally satisfy:

$$
\phi \odot_i A^{*_j} \leq (\phi \odot_i A)^{*_j}, \qquad \text{and} \qquad \text{(resp. } A^{*_j} \odot_i \phi \leq (A \odot_i \phi)^{*_j}\text{)},
$$

  for all $A \in K$, $\phi \in K_j$ and $0 \leq i < j < n$.

Globularity and modalities

- A modal n-Kleene algebra K is globular if

$$d_i \circ d_j = d_i \text{ and } d_i \circ r_j = d_i, \qquad d_j(A \odot_i B) = d_j(A) \odot_i d_j(B),$$
$$r_i \circ d_j = r_i, \text{ and } r_i \circ r_j = r_i, \qquad r_j(A \odot_i B) = r_j(A) \odot_i r_j(B).$$

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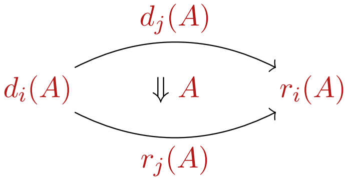
- An element $A \in K$ will be graphically represented by:



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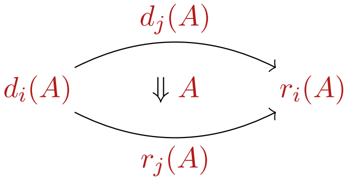
Globularity and modalities

- A modal n-Kleene algebra K is globular if

$$d_i \circ d_j = d_i \text{ and } d_i \circ r_j = d_i, \qquad d_j(A \odot_i B) = d_j(A) \odot_i d_j(B),$$
$$r_i \circ d_j = r_i, \text{ and } r_i \circ r_j = r_i, \qquad r_j(A \odot_i B) = r_j(A) \odot_i r_j(B).$$

for any $0 \leq i < j < n$.

- An element $A \in K$ will be graphically represented by:



for $i < j$.

- We define diamond operators

$$|A\rangle_j(\phi) = d_j(A \odot_j \phi), \quad \text{and} \quad \langle A|_j(\phi) = r_j(\phi \odot_j A),$$

where $A \in K$ and $\phi \in K_j$.
These are modal operators on the distributive lattice K_j.

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- Let K be a globular n-MKA and $0 \leq i < j < n$.

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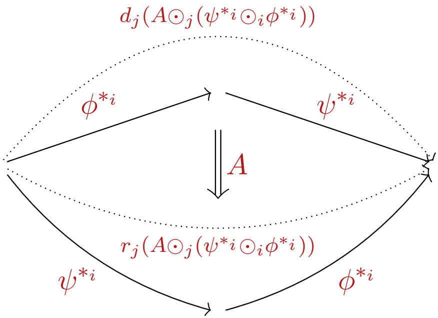
- This is graphically represented by



The Church-Rosser theorem

Church-Rosser in n-MKA

Theorem (Coherent Church-Rosser in globular n-MKA)

Let K be a globular n-MKA. Given $\phi, \psi \in K_j$, for $0 < j < n$, for any i-confluence filler $A \in K$ for (ϕ, ψ), we have

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq (\phi + \psi)^{*_i},$$

where $\hat{A} = (\phi + \psi)^{*_i} \odot_i A \odot_i (\phi + \psi)^{*_i}$.

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- Note that $|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \geq (\phi + \psi)^{*_i}$ is equivalent to

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- Furthermore, by restriction, we have

$$r_j\left(\hat{A}^{*_j} \odot_j (\psi^{*_i} \odot_i \phi^{*_i})\right) \leq \psi^{*_i} \odot_i \phi^{*_i}.$$

Proof

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- By the left i-star induction axiom, we have

$$1_i + (\phi + \psi)|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$$
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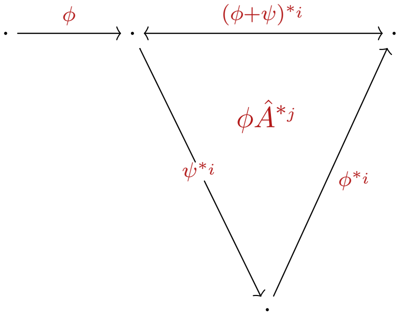
- By distributivity, we may prove this for each of the summands:

$$\phi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}), \\ \psi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}).$$

Proof

- In the case of ϕ:

$$\phi|\hat{A}^{*j}\rangle_j(\psi^{*i}\phi^{*i}) \leq |\phi\hat{A}^{*j}\rangle_j(\phi\psi^{*i}\phi^{*i})$$



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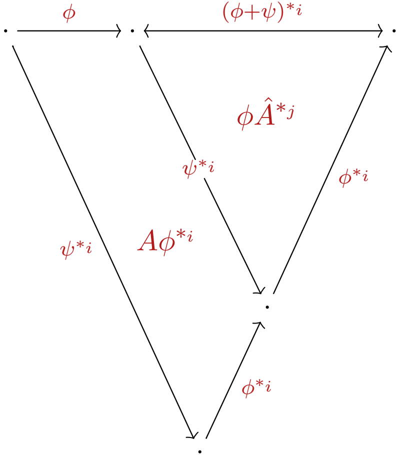

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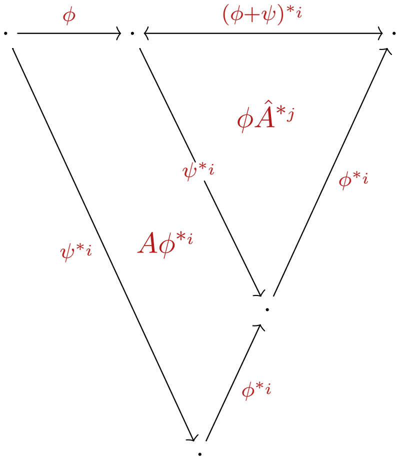

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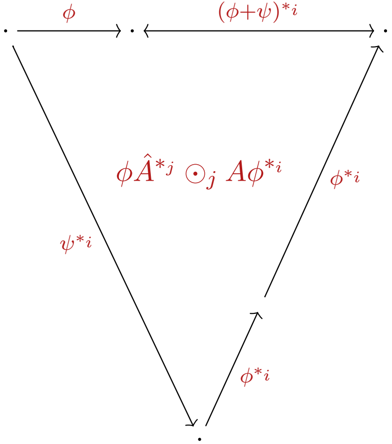

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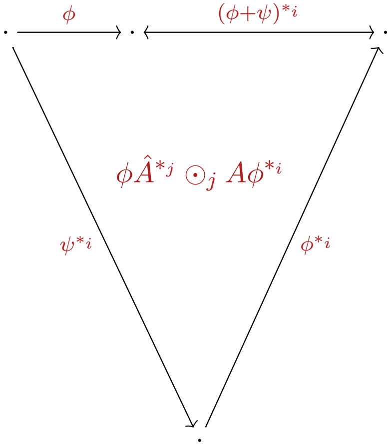

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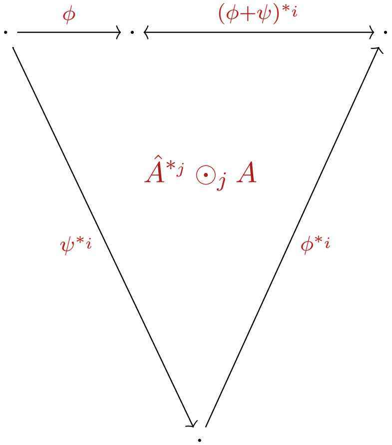

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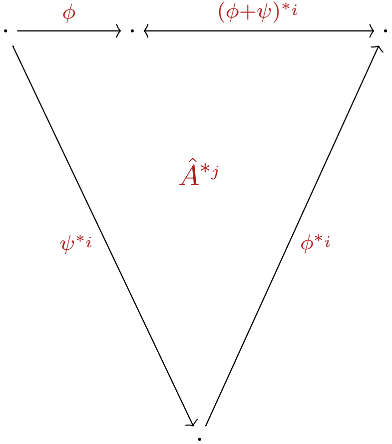

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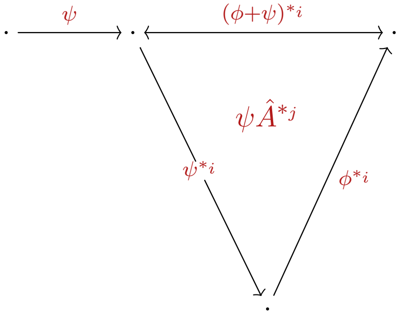

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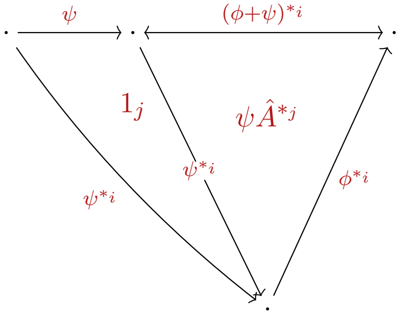

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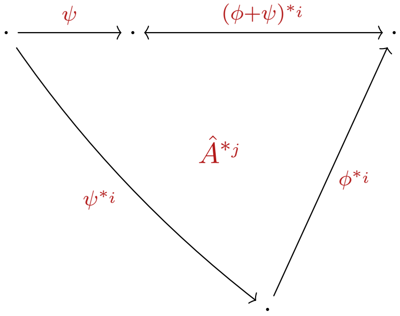

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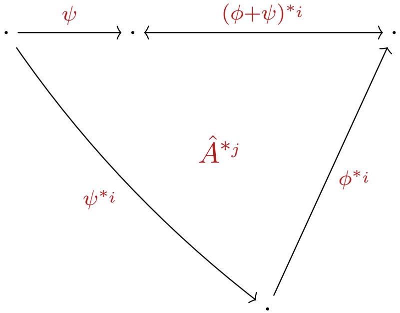

- This concludes the proof. □

Conclusion

Newman's Lemma

Theorem (Coherent Newman's lemma for globular p-Boolean MKA)

Let K be a globular p-Boolean MKA, and $0 \leq i \leq p < j < n$, such that

1. $(K_i, +, 0, \odot_i, 1_i, \neg_i)$ *is a complete Boolean algebra,*
2. *For all $\psi, \psi' \in K_j$, for all $(p_\alpha)_{\alpha \in I} \in K_i^I$ such that $sup_I(p_\alpha)$ exists,*

$$\psi \odot_i sup_I(p_\alpha) \odot_i \psi' = sup_I(\psi \odot_i p_\alpha \odot_i \psi').$$

If $\phi, \psi \in K_j$ are such that ψ is i-Noetherian and ϕ is i-well-founded, and if A is a local i-confluence filler for (ϕ, ψ), we have

$$|\hat{A}^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}) \geq \phi^{*_i}\psi^{*_i}.$$

Perspectives

- Recall Squier's theorem:

Theorem (Squier's theorem for ARS)

Let P be a n-polygraph and Γ a cellular extension of $P_n^\top$. If P is locally Γ-confluent, then for every pair (f,g) of parallel elements of $P_n^\top$ there exists $\alpha \in P_n^\top(\Gamma)$ with

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- Objective: Formulate and prove Squier's theorem in the setting of globular modal n-Kleene algebras.
- Provide an algebraic characterisation of cofibrance.

Thank you