Algebraic critical pair lemma

Cyrille Chenavier - Benjamin Dupont - Philippe Malbos

Institut Camille Jordan, Université Lyon 1

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Outline

I. Introduction : string and linear critical pair lemma

II. Algebraic polygraphs modulo

III. Algebraic critical pair lemma

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Algebraic rewriting and critical branching lemma

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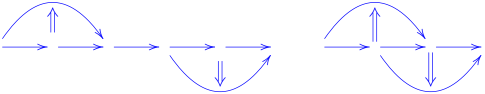

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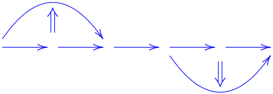

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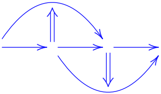

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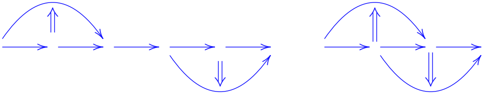

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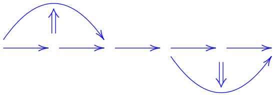

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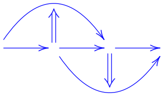

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- **Theorem (String critical pair lemma)** An SRS is locally confluent if and only if all its critical branchings are confluent.

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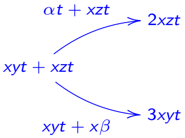

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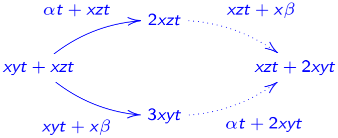

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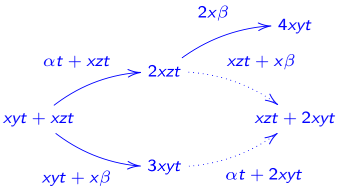

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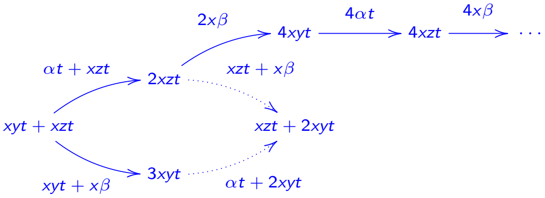

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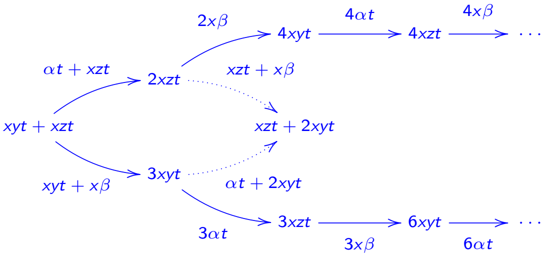

- CBL requires an additional termination assumption to hold.

II. Algebraic polygraphs modulo

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- **Example :** $P = \text{Mon}$, $Q = \{s,\ t\}$ and $R = \{\alpha\ :\ \mu(\mu(s,t),s) \Rightarrow \mu(t,\mu(s,t))\}$.

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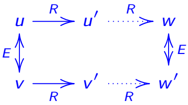

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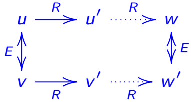

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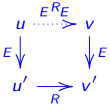

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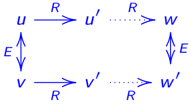

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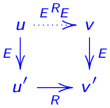

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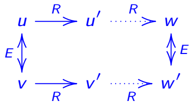

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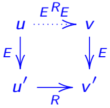

  - Rewriting system modulo : (R, E, S) such that $R \subseteq S \subseteq {}_E R_E$, Jouannaud-Kirchner '84.

- Algebraic polygraph modulo : quadruple (P, Q, R, S) where (P, Q, R) is an algebraic polygraph and S is a set of oriented relations such that

$$R \subseteq S \subseteq {}_{P_2\langle Q\rangle} R_{P_2\langle Q\rangle} := {}_P R_P.$$

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and CMod_2 contains the following generating 2-cells :

$x + 0 \Rrightarrow x$	(ring_1)	$x + (-x) \Rrightarrow 0$	(ring_2)
$-0 \Rrightarrow 0$	(ring_3)	$-(-x) \Rrightarrow x$	(ring_4)
$-(x + y) \Rrightarrow (-x) + (-y)$	(ring_5)	$x \cdot (y + z) \Rrightarrow x \cdot y + x \cdot z$	(ring_6)
$x \cdot 0 \Rrightarrow 0$	(ring_7)	$x \cdot (-y) \Rrightarrow -(x \cdot y)$	(ring_8)
$1 \cdot x \Rrightarrow x$	(ring_9)	$a \oplus 0^{\oplus} \Rrightarrow a$	(mod_1)
$x.(y.a) \Rrightarrow (x \cdot y).a$	(mod_2)	$1.a \Rrightarrow a$	(mod_3)
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- **Theorem [Peterson-Stickel, Hullot]** CMod is a presentation of the theory of modules over commutative rings that is confluent modulo AC.

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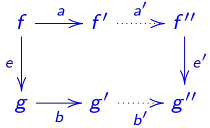

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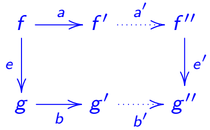

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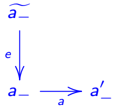

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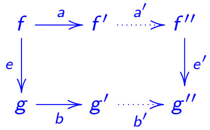

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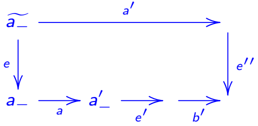

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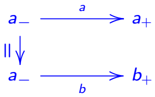

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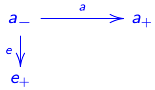

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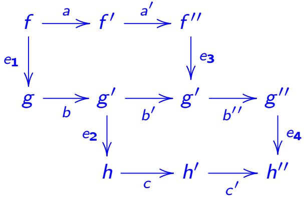
- When ${}_{P_2\langle Q\rangle}R \subseteq S$, property $\mathbf{b_0}$) is always satisfied.

III. Algebraic critical branching lemma

Algebraic rewriting systems and critical branching lemma

- Confluence modulo diagrams of an APM :



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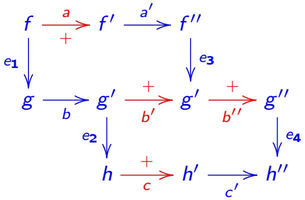

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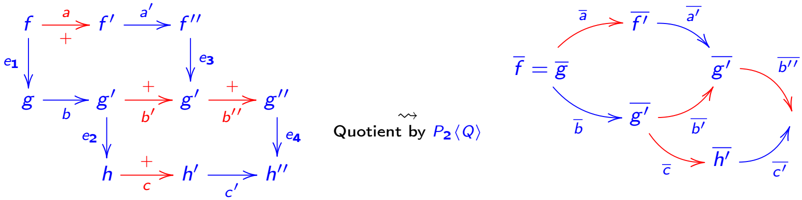

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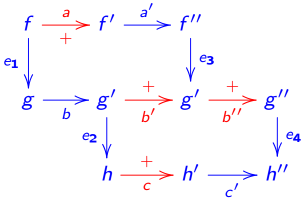
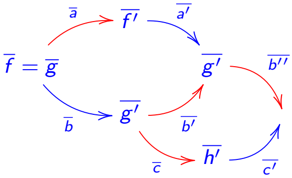


$\rightsquigarrow$ **Quotient by** $P_2\langle Q\rangle$



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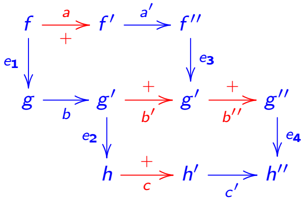

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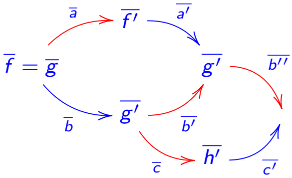

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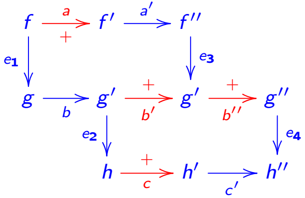

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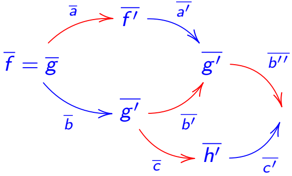

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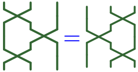
$\langle s,\ t \mid sts \to tst\ \rangle$ $s =$ , $t =$ , 

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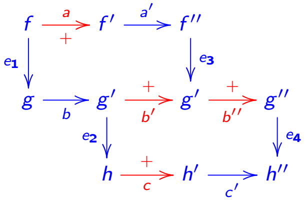

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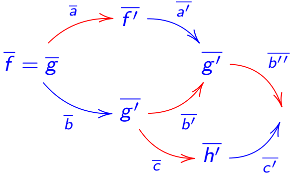

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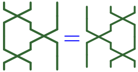
$\langle s,\ t \mid sts \to tst\ \rangle$ $s =$ , $t =$ , 

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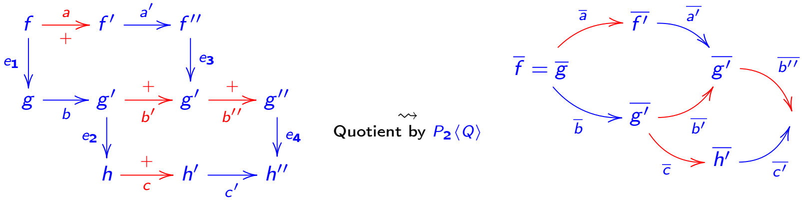

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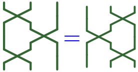
$\langle s,\ t \mid sts \to tst \ \rangle$ $s =$ , $t =$ , 

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- **Theorem [Chenavier - D. - Malbos]** Let $\mathcal{P} = (P, Q, R, S)$ be a quasi-terminating and positively σ-confluent APM, and $\mathcal{A}$ be an ARS on $\mathcal{P}$. Then $\mathcal{A}$ is locally confluent if and only if its critical branchings are confluent.

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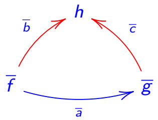

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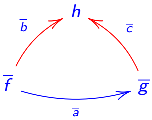

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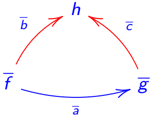

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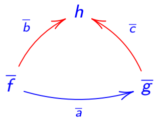

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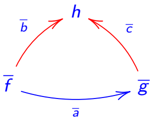

- **Questions :**
  - Is the positive σ-confluence always satisfied in the linear setting ?
  - How does the critical branching lemma translates if we change the positive strategy ?
- **Conclusion :**
  - This work suggests new tools for rewriting in various algebraic structures.
  - Need a better understanding of how to choose strategies, and ensure positive confluence in general.
  - Develop a critical branching lemma for various algebraic contexts : groups, differential algebras, operads, higher-dimensional categories.

Thank you !