Parallel Closedness Revisited

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joint work with

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June 30, 2020

About This Talk

1. ~~new~~ **proof of Huet's parallel closedness**

2. **remark on Liu and Jouannaud's work**

3. **comparison of closedness results**
   Huet 1980, Toyama 1981, Toyama 1988, Gramlich 1996

Parallel Closedness

Theorem (Huet 1980)

left-linear TRS is confluent if $\leftarrow\rtimes\xrightarrow{\epsilon}\ \subseteq\ \twoheadrightarrow\!\!\!\!\!\!+\ $

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left-linear TRS is confluent if $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\parallel\!\!\!\rightarrow}$

Notation

$t \leftarrow\!\rtimes\!\xrightarrow{\epsilon} u$ denotes critical pair (t, u)

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Theorem (Huet 1980)

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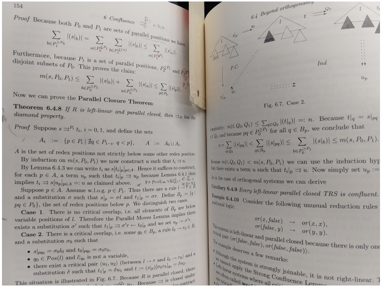
Notation

$t \leftarrow\!\rtimes\!\xrightarrow{\epsilon} u$ denotes critical pair (t, u)

Definition ($\mathrel{+\!\!\!+\!\!\!\rightarrow}$)

- $x \overset{\varnothing}{\mathrel{+\!\!\!+\!\!\!\rightarrow}} x$ for all variables x,
- $f(s_1, \ldots, s_n) \overset{P}{\mathrel{+\!\!\!+\!\!\!\rightarrow}} f(t_1, \ldots, t_n)$ if
  $s_i \overset{P_i}{\mathrel{+\!\!\!+\!\!\!\rightarrow}} t_i$ and $P = \{i \cdot p \mid 1 \leqslant i \leqslant n \text{ and } p \in P_i\}$, and
- $\ell\sigma \overset{\{\epsilon\}}{\mathrel{+\!\!\!+\!\!\!\rightarrow}} r\sigma$ if $\ell \to r \in \mathcal{R}$

Textbook Proofs



154

6 Confluence

Proof Because both P_0 and P_1 are sets of parallel positions we have

$$\sum_{b \in P_1^{\geq P_0}} |(s|_b)| = \sum_{a \in P_0^{\leq P_1}} \sum_{b \in P_1^{\geq (a)}} |(s|_b)| \leq \sum_{a \in P_0^{\leq P_1}} |(s|_a)|.$$

Furthermore, because P_1 is a set of parallel positions, $P_0^{\leq P_1}$ and $P_0^{\geq P_1}$ … disjoint subsets of P_0. This proves the claim:

$$m(s, P_0, P_1) \leq \sum_{b \in P_0^{\geq P_1}} |(s|_b)| + \sum_{a \in P_0^{\leq P_1}} |(s|_a)| \leq \sum_{p \in P_0} |(s|_p)|.$$

Now we can prove the **Parallel Closure Theorem**:

Theorem 6.4.8 *If R is left-linear and parallel closed, then $\rightrightarrows_R$ has the diamond property.*

Proof Suppose $s \rightrightarrows^{P_i} t_i$, $i = 0, 1$, and define the sets

$$A_i := \{p \in P_i \mid \nexists q \in P_{1-i}.\ q < p\}, \qquad A := A_0 \cup A_1.$$

A is the set of redex positions not strictly below some other redex position.

By induction on $m(s, P_0, P_1)$ we now construct u such that $t_i \rightrightarrows u$.

By Lemma 6.4.3 we can write t_i as $s[t_i|_p]_{p \in A}$. Hence it suffices to construct, for each $p \in A$, a term u_p such that $t_i|_p \rightrightarrows u_p$ because Lemma 6.4.1 then implies $t_i \rightrightarrows s[u_p]_{p \in A} =: u$ as claimed above.

Suppose $p \in A$. Assume w.l.o.g. $p \in P_1$. Thus there are a rule $l \to r \in R$ and a substitution σ such that $s|_p = \sigma l$ and $t_1|_p = \sigma r$. Define $B_p := \{q \mid pq \in P_0\}$, the set of redex positions below p. We distinguish two cases.

Case 1. There is no critical overlap, i.e. all elements of B_p are below variable positions of l. Therefore the Parallel Moves Lemma implies there exists a substitution σ' such that $t_1|_p \rightrightarrows \sigma' r \leftarrow t_0|_p$ and we set $u_p := \sigma' r$.

Case 2. There is a critical overlap, i.e. some $q_0 \in B_p$, a rule $l_0 \to r_0 \in R$, and a substitution σ_0 such that

- $s|_{pq_0} = \sigma_0 l_0$ and $t_0|_{pq_0} = \sigma_0 r_0$,
- $q_0 \in \mathcal{P}os(l)$ and $l|_{q_0}$ is not a variable,
- there exist a critical pair $\langle u_1, u_2 \rangle$ (between $l \to r$ and $l_0 \to r_0$) and a substitution δ such that $t_1|_p = \delta u_1$ and $t := (s|_p)[\sigma_0 r_0]_{q_0} = \delta u_2$.

This situation is illustrated in Fig. 6.7. Because R is parallel closed, there …

6.4 Beyond orthogonality

Fig. 6.7. Case 2.

… complexity: $m(t, Q_0, Q_1) \leq \sum_{q \in Q_0} |(t|_q)| =: n$. Because $t|_q = s|_{pq}$ … $q \in Q_0$, and because $pq \in P_0^{\geq P_1}$ for all $q \in B_p$, we conclude that

$$n = \sum_{q \in Q_0} |(s|_{pq})| < \sum_{q \in B_p} |(s|_{pq})| \leq \sum_{b \in P_0^{\geq P_1}} |(s|_b)| \leq m(s, P_0, P_1).$$

Because $m(t, Q_0, Q_1) < m(s, P_0, P_1)$ we can use the induction hyp… that there exists a term u such that $t_i|_p \rightrightarrows u$. Now simply set $u_p :=$ …

As in the case of orthogonal systems we can derive

Corollary 6.4.9 *Every left-linear parallel closed TRS is confluent.*

Example 6.4.10 Consider the following unusual reduction rules … propositional logic:

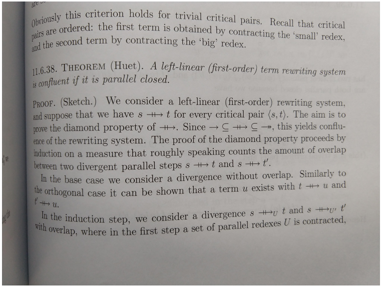
$$or(x, false) \to or(x, x),$$
$$or(false, y) \to or(y, y).$$

This system is left-linear and parallel closed because there is only one critical pair $\langle or(false, false), or(false, false) \rangle$.

The example deserves a few remarks:

- Although the system is strongly joinable, it is not right-linear. … we cannot apply the Strong Confluence Lemma …
- Left-linear systems where all critical … are often called … are trivial …

“Term Rewriting and All That”,
Baader and Nipkow, 1998



Obviously this criterion holds for trivial critical pairs. Recall that critical pairs are ordered: the first term is obtained by contracting the ‘small’ redex, and the second term by contracting the ‘big’ redex.

11.6.38. Theorem (Huet). *A left-linear (first-order) term rewriting system is confluent if it is parallel closed.*

Proof. (Sketch.) We consider a left-linear (first-order) rewriting system, and suppose that we have $s \twoheadrightarrow\!\!\!\!\!\!\,+\, t$ for every critical pair $\langle s, t \rangle$. The aim is to prove the diamond property of ⇸. Since $\to \subseteq$ ⇸ $\subseteq \twoheadrightarrow$, this yields confluence of the rewriting system. The proof of the diamond property proceeds by induction on a measure that roughly speaking counts the amount of overlap between two divergent parallel steps s ⇸ t and s ⇸ t'.

In the base case we consider a divergence without overlap. Similarly to the orthogonal case it can be shown that a term u exists with t ⇸ u and t' ⇸ u.

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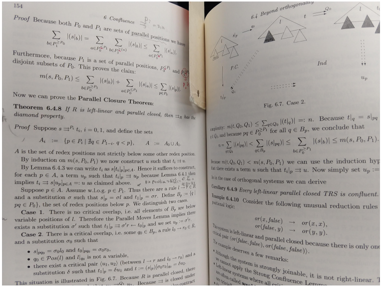

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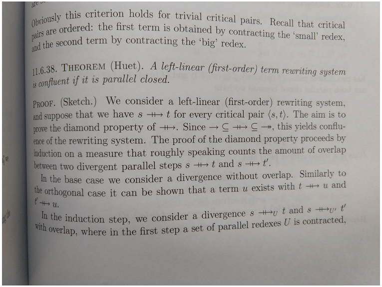

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Definition (Huet 1980, Baader and Nipkow 1998)

$|s, P, Q| = \Sigma_{p \in P_0} |(s|_p)| + \Sigma_{q \in Q_0} |(s|_q)|$ where

- $P_0 = \{p \mid p \geqslant q \text{ for some } q \in Q\}$, and
- $Q_0 = \{q \mid q > p \text{ for some } p \in P\}$

Example of Huet's Measure

consider left-linear TRS (COPS #35)

$$\mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \qquad \mathsf{f}(x,\mathsf{b}) \to \mathsf{g}(\mathsf{f}(x,x))$$

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{f}(\mathsf{b},x) \to \mathsf{g}(\mathsf{f}(x,x))$$

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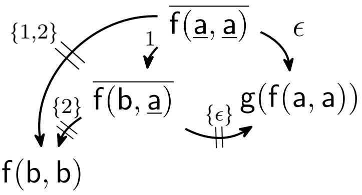
parallel peak:



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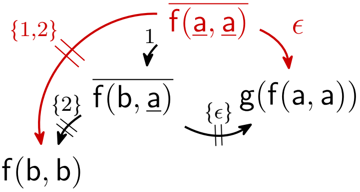

$$|\overline{\mathsf{f}(\underline{\mathsf{a}},\underline{\mathsf{a}})}, \{1,2\}, \{\epsilon\}| = \quad > |\overline{\mathsf{f}(\mathsf{b},\underline{\mathsf{a}})}, \{2\}, \{\epsilon\}| =$$

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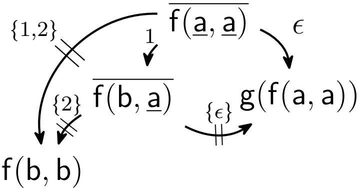

$$|\overline{\mathsf{f}(\underline{\mathsf{a}},\underline{\mathsf{a}})},\{1,2\},\{\epsilon\}| = {\color{red}2} > |\overline{\mathsf{f}(\mathsf{b},\underline{\mathsf{a}})},\{2\},\{\epsilon\}| =$$

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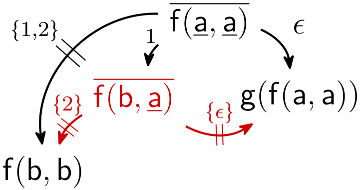

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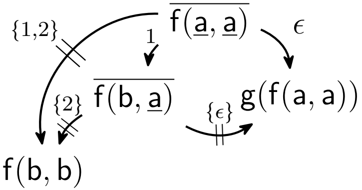

$$|\overline{\mathsf{f}(\underline{\mathsf{a}},\underline{\mathsf{a}})},\{1,2\},\{\epsilon\}| = 2 > |\overline{\mathsf{f}(\mathsf{b},\underline{\mathsf{a}})},\{2\},\{\epsilon\}| = 1$$

Huet's Proof of Parallel Closedness

Theorem (Huet 1980)

left-linear TRS is confluent if $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!\!\parallel\!\!\!\rightarrow}$

Huet's Proof of Parallel Closedness

Theorem (Huet 1980)

left-linear TRS is confluent if $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \twoheadrightarrow\!\!\!\!\!+$

Proof

$\twoheadrightarrow\!\!\!\!\!+$: $\diamond$ by induction on $(|s, P, Q|, s)$ wrt $(>, \rhd)$

Huet's Proof of Parallel Closedness

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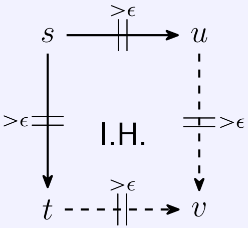

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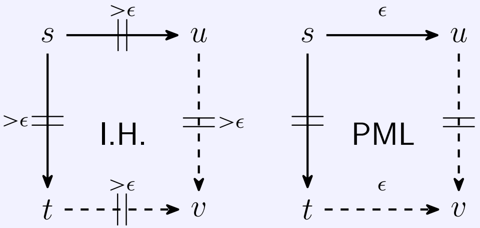

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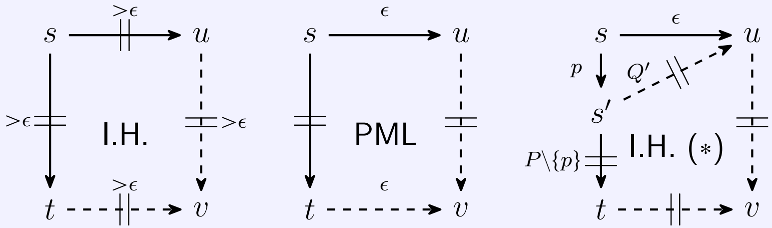

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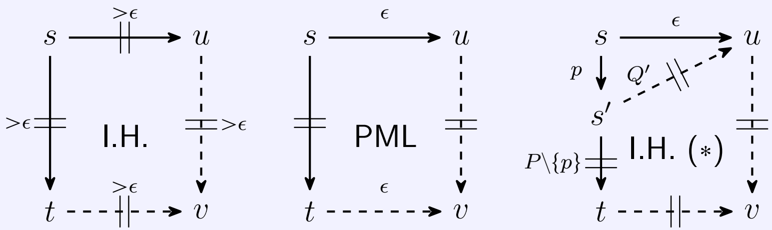

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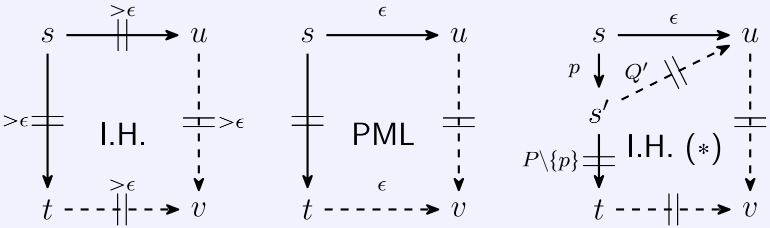


$(*)$ $|s, P, \{\epsilon\}| > |s', P\backslash\{p\}, Q'|$ $\Leftarrow$ **difficult!!**

Ingenious Weight

Baader and Nipkow 1998

The Parallel Closure Theorem relies on an ingenious induction to reduce multiple overlaps to critical pairs

Nagele 2017

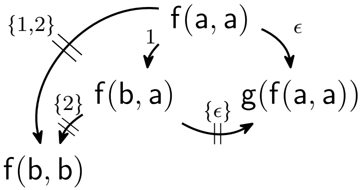
Consequently, when starting the present formalization, we also adopted this definition. However, the book keeping required by working with sets of positions as well as formally reasoning about this measure in Isabelle became so convoluted that it very much obscured the ingenuity and elegance of Huet's original idea while at the same time defeating our formalization efforts.

New Induction Measure

Definition (inspired by Oyamaguchi and Ohta 1997)

$|t|_P = \Sigma_{p \in P} |(t|_p)|$

Example



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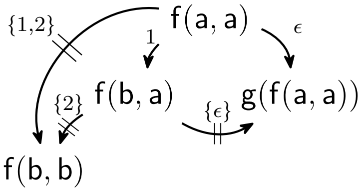

$$|\mathsf{f}(\mathsf{b},\mathsf{b})|_{\{1,2\}} =$$
$$|\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a}))|_{\{\epsilon\}} =$$
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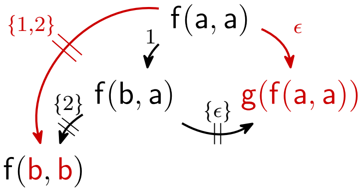

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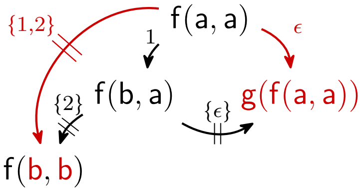

$|\mathsf{f}(\mathsf{b},\mathsf{b})|_{\{1,2\}} = 2$

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New Induction Measure

Definition (inspired by Oyamaguchi and Ohta 1997)

$|t|_P = \Sigma_{p \in P} |(t|_p)|$

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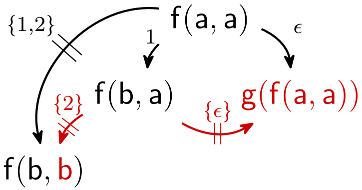

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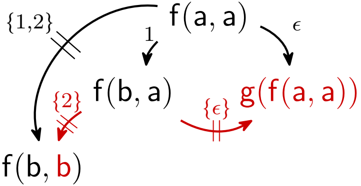

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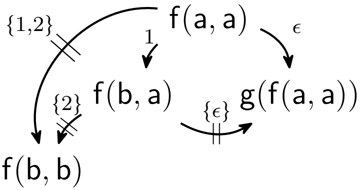

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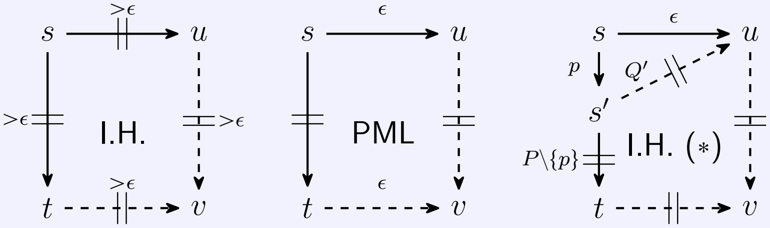
Lemma

- $|t|_{\{\epsilon\}} \geqslant |t|_P$ *if* $P \subseteq \mathcal{P}\mathsf{os}(t)$ *and* P *is parallel*
- $|t|_P > |t|_{P'}$ *if* $P' \subsetneq P$

New Proof of Huet's Parallel Closedness

Proof.

$\twoheadrightarrow\!\!\!\!\!\!+$: $\diamond$ by induction on $(|t|_P + |u|_Q, s)$ wrt $(>, \rhd)_{lex}$



(∗) $|t|_P + |u|_{\{\epsilon\}} > |t|_{P\setminus\{p\}} + |u|_{Q'}$ □

About This Talk

1 ~~new~~ proof of Huet's parallel closedness

2 **remark on Liu and Jouannaud's work**

3 comparison of closedness results
Huet 1980, Toyama 1981, Toyama 1988, Gramlich 1996

Almost Parallel Closedness

	H80
YES	36

* 437 left-linear TRSs from COPS

Theorem (parallel closedness, Huet 1980)

left-linear TRS is confluent if $\leftarrow\rtimes\xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!\!\!\!\longrightarrow}$

Almost Parallel Closedness

	H80	T88
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Theorem (almost parallel closedness, Toyama 1988)

left-linear TRS is confluent if

$$\xleftarrow{>\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!\!\!+\!\!\!\!\to} \;\&\; \xleftarrow{\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!\!\!+\!\!\!\!\to} \cdot \overset{*}{\leftarrow}$$

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Theorem (parallel closedness, Huet 1980)

left-linear TRS is confluent if $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!\shortparallel\!\!\!\rightarrow}$

Theorem (almost parallel closedness, Toyama 1988)

left-linear TRS is confluent if

$\xleftarrow{>\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!\shortparallel\!\!\!\rightarrow}$ & $\xleftarrow{\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\!\shortparallel\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow$

Toyama 1988 subsumes Huet 1980

van Oostrom (June 16, 2020, private communication)

liu's proof may look quite different to your proof, but it seems based on a similar main idea

Jiaxiang Liu and Jean-Pierre Jouannaud

Confluence: The Unifying, Expressive Power of Locality

Specification, Algebra, and Software, LNCS 8373, pp. 337–358, 2014

https://doi.org/10.1007/978-3-642-54624-2_17

Liu and Jouannaud's Proof of Toyama 1988

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Lemma 13 of Liu and Jouannaud 2014.

commutation of $\rightarrow$ and $\mathrel{+\!\!\!\!\!\!\rightarrow}$ is shown by Liu and Jouannaud's decreasing diagram (Theorem 5) with labels

1. $\mathrel{+\!\!\!\!\!\!\rightarrow}_{(1,0)}$ for all parallel steps, and
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- **our proof is essentially same as Liu and Jouannaud's**
- **this result should be attributed to Liu and Jouannaud**

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Gramlich's Criterion based on Parallel Critical Pair

	H80	T88	G96
YES	36	49	**56**

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Theorem (Gramlich 1996)

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Notation

$t \overset{P}{\mathrel{\leftarrow\!\!\!+\!\!\!\!+}}\!\rtimes\!\rightarrow u$ denotes parallel critical pair

Gramlich 1996 Does Not Subsume Huet 1980

consider left-linear parallel closed TRS (COPS #35) again

$$\mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \qquad \mathsf{f}(x,\mathsf{b}) \to \mathsf{g}(\mathsf{f}(x,x))$$

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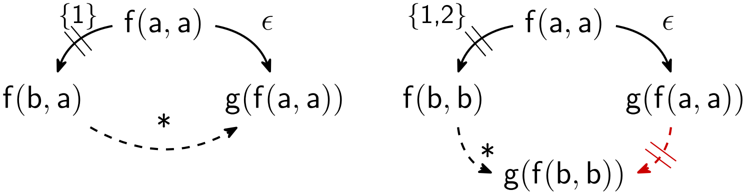

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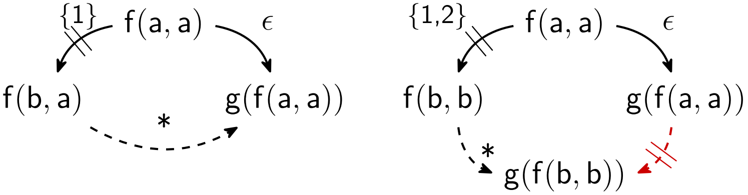

TRS does not satisfy Gramlich's conditions

Toyama's Parallel Critical Pair

	H80	T88	G96
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Theorem (Toyama 1981)

left-linear TRS is confluent

- $\xleftarrow{>\epsilon} \rtimes \xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!\!\!+\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow$ &
- *for every parallel critical peak* $t \xleftarrow{P}\!\!\!\!\!+\!\!\!+ s \xrightarrow{\epsilon} u$
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Example for Variable Condition

remark

for every parallel critical peak $t \overset{P}{\leftarrow\!\!\!+\!\!\!+\!\!\!-} s \xrightarrow{\epsilon} u$
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$$\mathsf{f}(\mathsf{g}(x), y) \to \mathsf{f}(\mathsf{h}_2(x), y) \qquad \mathsf{g}(x) \to \mathsf{h}_1(x) \quad \mathsf{h}_2(x) \to x$$
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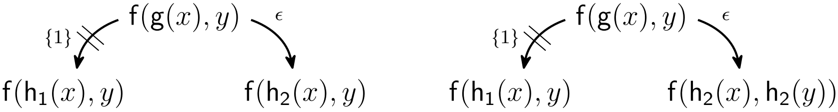

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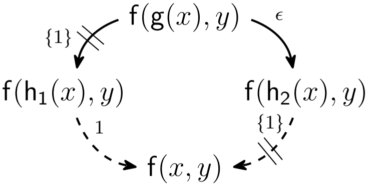

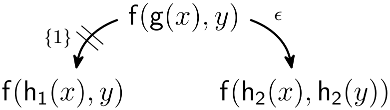

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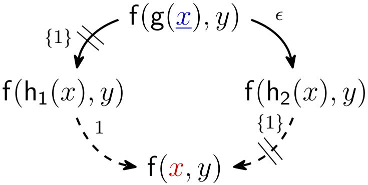

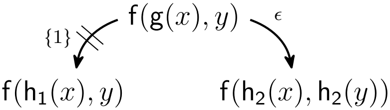

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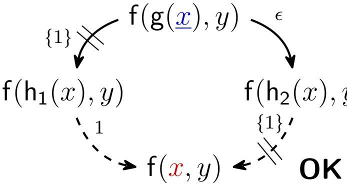

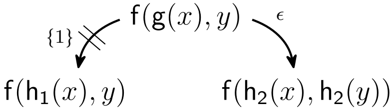

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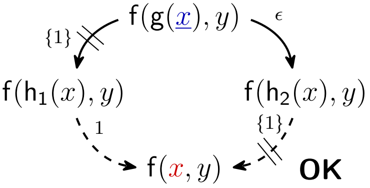
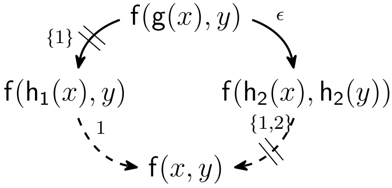


OK



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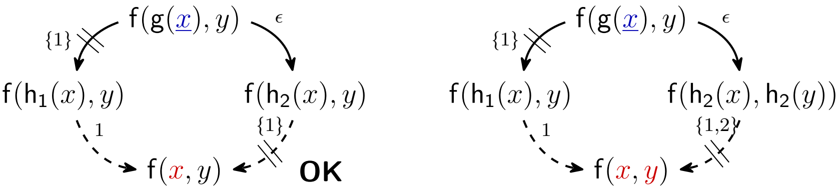

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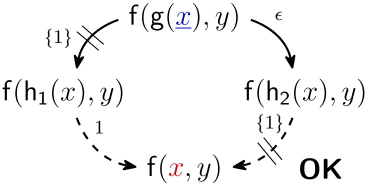

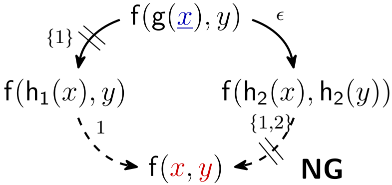

Example of Toyama 1981

consider left-linear parallel closed TRS again

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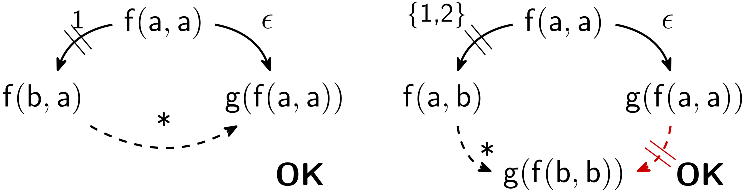
parallel critical pairs:



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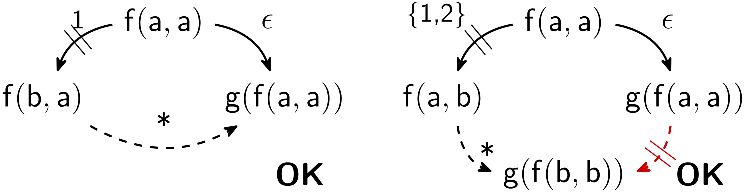

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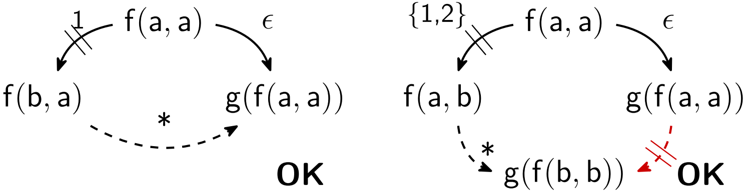


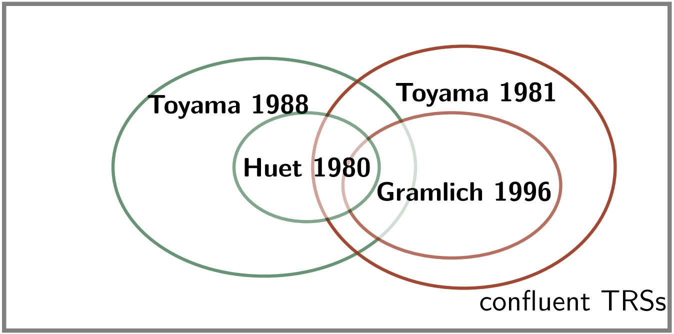
TRS satisfies Toyama 1981's conditions

Q: does Toyama 1981 subsume Toyama 1988?

Known Results

Baader and Nipkow, 1998: bibliographic notes in Chapter 6

An alternative and **incomparable** approach based on parallel critical pairs is due to Toyama. Exercise 6.22 is based on the work of Gramlich who rediscovered a slightly restricted version of Toyama's main theorem.



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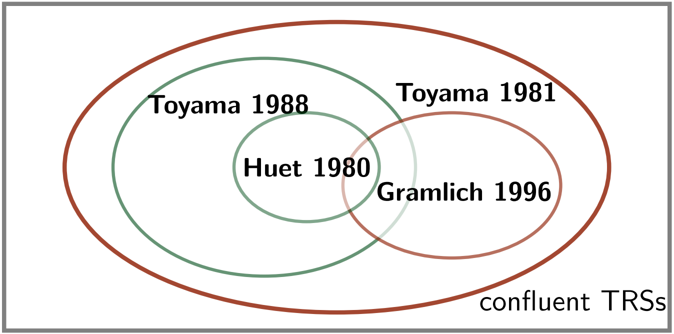

Toyama 1981 Subsumes Toyama 1988

Lemma

for every left-linear almost parallel closed TRS

if $t \xleftarrow{P_1}\!\!\!\!\!\!+\!\!\!\!+ \; s \xrightarrow{P_2}\!\!\!\!\!\!+\!\!\!\!+ \; u$ *then there exist* v_1, v_2, Q_1, Q_2 *such that*

- $t \rightarrow^* v_1 \xleftarrow{Q_1}\!\!\!\!\!\!+\!\!\!\!+ \; u$ *and* $\mathcal{V}\mathsf{ar}(s, P_1) \supseteq \mathcal{V}\mathsf{ar}(v_1, Q_1)$, *and*
- $t \xrightarrow{Q_2}\!\!\!\!\!\!+\!\!\!\!+ \; v_2 \; {}^*\!\!\leftarrow u$ *and* $\mathcal{V}\mathsf{ar}(s, P_2) \supseteq \mathcal{V}\mathsf{ar}(v_2, Q_2)$

Toyama 1981 Subsumes Toyama 1988

Lemma

for every left-linear almost parallel closed TRS

if $t \overset{P_1}{\leftarrow\!\!\!\!+\!\!\!+} s \overset{P_2}{+\!\!\!+\!\!\!\!\rightarrow} u$ *then there exist* v_1, v_2, Q_1, Q_2 *such that*

- $t \rightarrow^* v_1 \overset{Q_1}{\leftarrow\!\!\!\!+\!\!\!+} u$ *and* $\mathcal{V}\mathrm{ar}(s, P_1) \supseteq \mathcal{V}\mathrm{ar}(v_1, Q_1)$, *and*
- $t \overset{Q_2}{+\!\!\!+\!\!\!\!\rightarrow} v_2 \ {}^*\!\!\leftarrow u$ *and* $\mathcal{V}\mathrm{ar}(s, P_2) \supseteq \mathcal{V}\mathrm{ar}(v_2, Q_2)$

Proof.

by induction $(|t|_{P_1} + |u|_{P_2}, s)$ wrt $(>, \rhd)_{lex}$

Toyama 1981 Subsumes Toyama 1988

Lemma

for every left-linear almost parallel closed TRS

if $t \xleftarrow{P_1}\!\!\!\!\!\!\!+\!\!\!+ \; s \xrightarrow{P_2}\!\!\!\!\!\!\!+\!\!\!+ \; u$ *then there exist* v_1, v_2, Q_1, Q_2 *such that*

- $t \rightarrow^* v_1 \xleftarrow{Q_1}\!\!\!\!\!\!\!+\!\!\!+ \; u$ *and* $\mathcal{V}\mathsf{ar}(s, P_1) \supseteq \mathcal{V}\mathsf{ar}(v_1, Q_1)$*, and*
- $t \xrightarrow{Q_2}\!\!\!\!\!\!\!+\!\!\!+ \; v_2 \; {}^*\!\!\leftarrow u$ *and* $\mathcal{V}\mathsf{ar}(s, P_2) \supseteq \mathcal{V}\mathsf{ar}(v_2, Q_2)$

Proof.

by induction $(|t|_{P_1} + |u|_{P_2}, s)$ wrt $(>, \rhd)_{lex}$

Theorem

Toyama 1981 subsumes Toyama 1988

Conclusion

- simple proof for parallel closedness
  but it turned out this is reproduction of
  Liu and Jouannaud's proof
- criterion of Toyama 1981 subsumes others
  c.f. Okui 1998 subsumes van Oostrom 1997