Confluence of Drag Rewriting

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Outline

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Drags, Terms and Graphs

- Drags are a generalization of terms
- Drags are a special kind of labelled graphs with variables

Confluence of Graph Rewriting

- Confluence for terminating graph rewriting systems is undecidable [Plump 1993]

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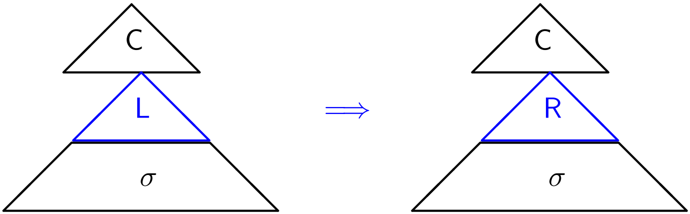
- Confluence for terminating graph rewriting systems is undecidable [Plump 1993]
- Confluence for terminating graph rewriting systems with interfaces is decidable [Bonchi et al 2017]
- Confluence for Drags rewriting systems is decidable

Outline

Tree Rewriting and Composition



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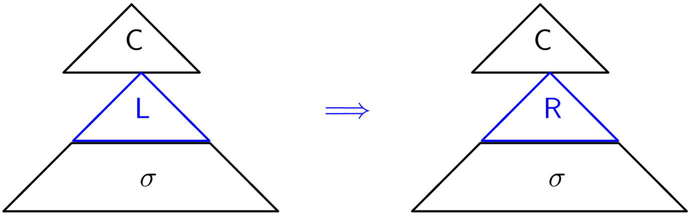

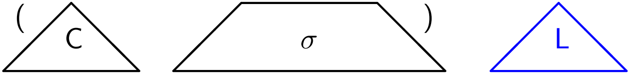

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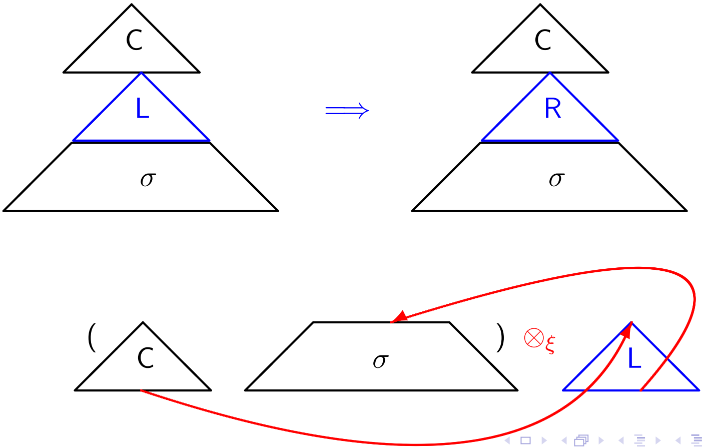

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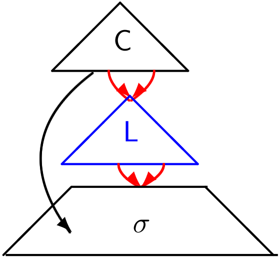

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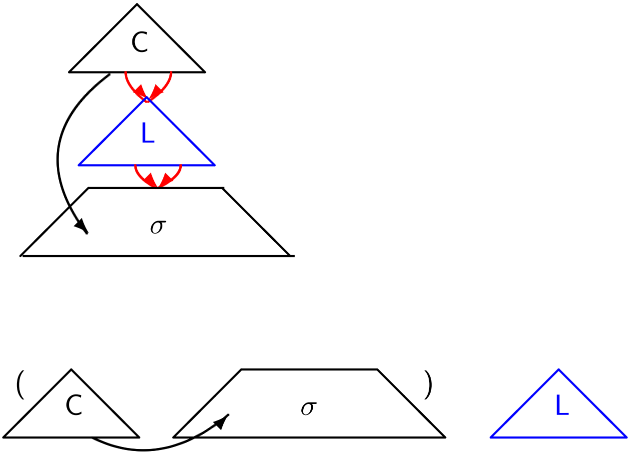

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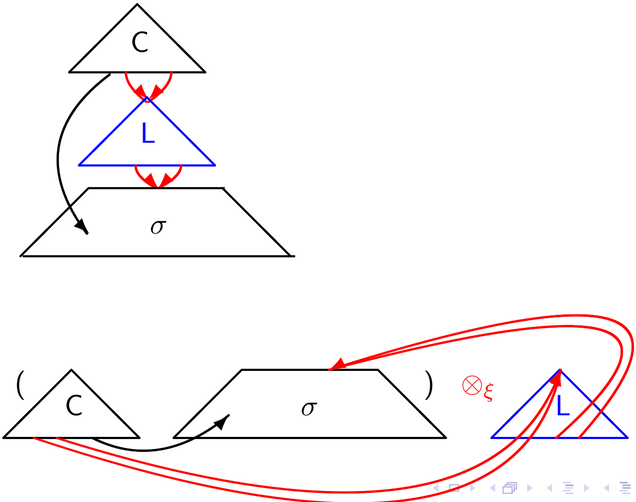

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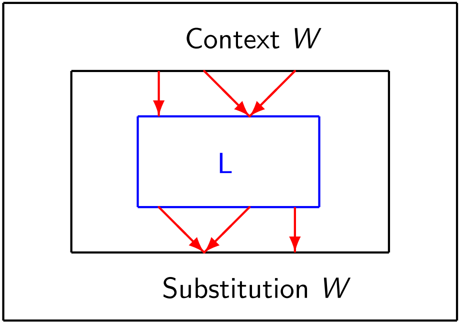

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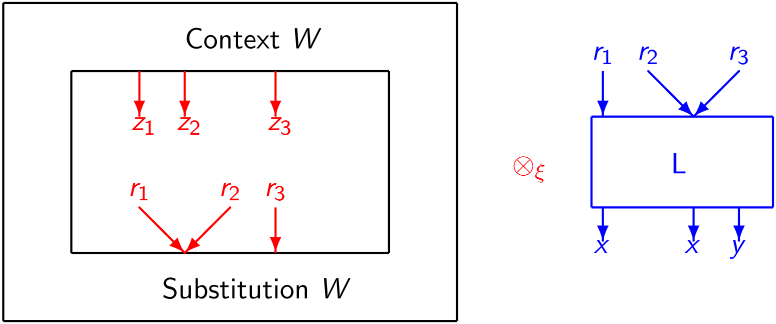


Linear context, non-linear left-hand side

Outline

Drags

We presuppose two sets Σ (function symbols) and Ξ (variables) whose elements have a fixed arity. Variables have arity 0.

Definition

A *drag* is a tuple $\langle V, R, S, L, X \rangle$, where

1. V is the finite set of *vertices*;
2. $R : [1..|R|] \to V$ is a finite list of vertices, called *roots*;
3. $S \subseteq V$ is the set of *sprouts*, $V \setminus S$ is the set of *internal* vertices;
4. $L : V \to \Sigma \cup \Xi$ is the *labeling* function, mapping internal vertices to labels from the vocabulary Σ and sprouts to labels from the vocabulary Ξ;
5. $X : V \to V^*$ is the *successor* function, mapping each vertex $v \in V$ to a list of vertices in V such that $|X(v)| = |L(v)|$.

Drag Composition uses a Switchboard

Definition

Let $D = \langle V, R, L, X, S \rangle$ and $D' = \langle V', R', L', X', S' \rangle$ be drags. A *switchboard* ξ for (D, D') is a pair $\langle \xi_D : S \to [1 \,..\, |R'|], \xi_{D'} : S' \to [1 \,..\, |R|] \rangle$ of partial injective functions (we also say that (D', ξ) is an *extension* of D), such that

(i) $\forall s, t \in S$ such that $s \in \mathcal{D}\wr\Updownarrow_{\xi_D}$ and $L(s) = L(t)$, then $t \in \mathcal{D}\wr\Updownarrow_{\xi_D}$ and $R'(\xi_D(s)) = R'(\xi_D(t))$;

(ii) $\forall s, t \in S'$ such that $s \in \mathcal{D}\wr\Updownarrow_{\xi_D}$ and $L'(s) = L'(t)$, then $t \in \mathcal{D}\wr\Updownarrow_{\xi_D}$ and $R(\xi_{D'}(s)) = R(\xi_{D'}(t))$;

(iii) $\xi*$ does not induce cycles among the sprouts of the two drags.

Rewriting Switchboard

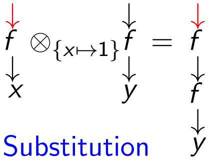
Definition (Rewriting switchboard and extension)

ξ is a *rewriting switchboard* for (D, L) if

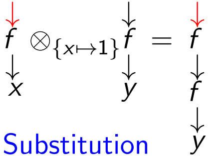
- ξ_D is linear and *surjective* (all roots of L must disappear)
- ξ_L is *total* (all sprouts must disappear)

(D, ξ) is called a *rewriting extension* of L.

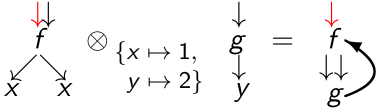
Examples of composition



Examples of composition

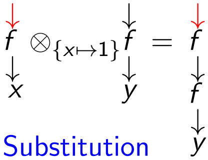


Substitution



Forced sharing: a choice

Examples of composition



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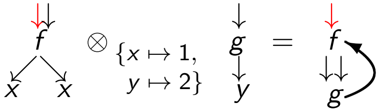
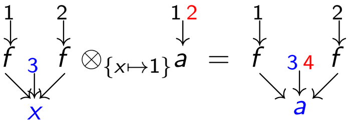
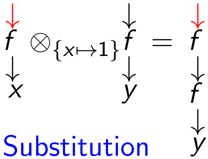


Forced sharing: a choice



Root transfer

Examples of composition



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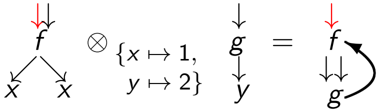

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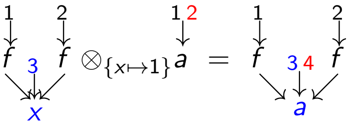

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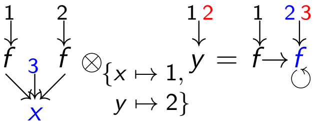

Double root transfer

Outline

Drag unification versus tree unification

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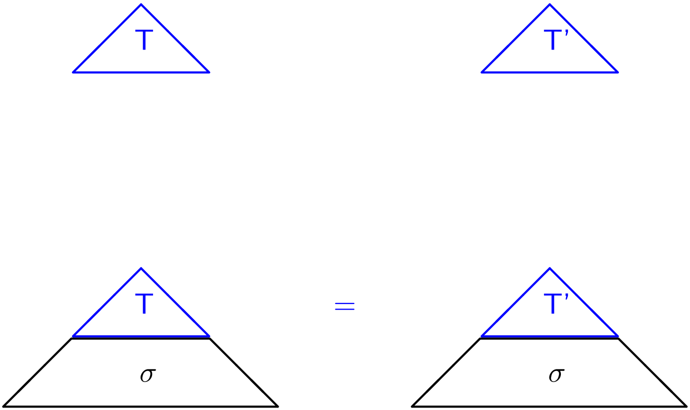
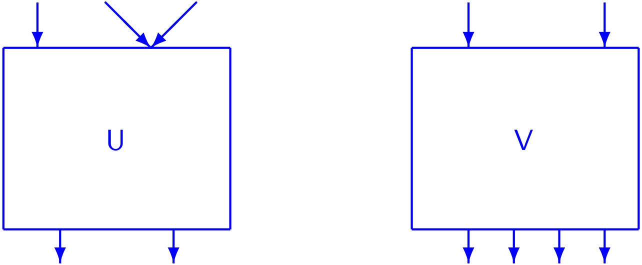
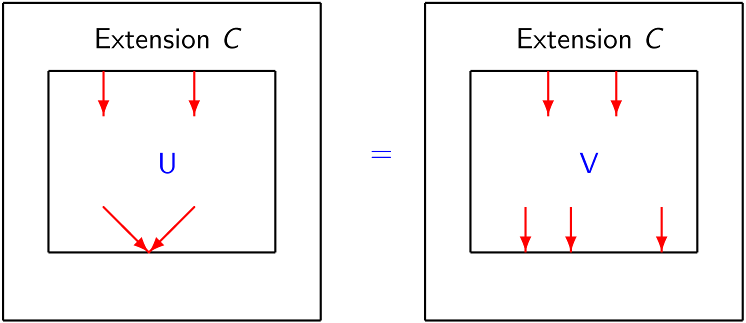


σ is a unifier of D and D'

Drag unification versus tree unification



Drag unification versus tree unification



C is a unifier of U and V

Unifiers

Definition

Given

- two drags U, V that share no vertex names nor root numbers,
- two lists of internal *partner vertices* $\overline{u} \in U, \overline{v} \in V$ of equal length,

the *unification problem* $U[\overline{u}] = V[\overline{v}]$ admits as *solutions* or *unifiers*, the common extensions of both U and V which identify $U \otimes_\xi C$ and $V \otimes_\xi C$ below $\overline{u}$ and $\overline{v}$ respectively, that is, the subdrags of $U \otimes_\xi C$ at $\overline{u}$ and of $V \otimes_\xi C$ at $\overline{v}$ must be isomorphic, that is, identical up to vertex names.

Subsumption

Definition

We say that a drag U subsumes a drag V if there is a context extension (C, ξ) of U such that $V = C \otimes_\xi U$.

Lemma

Subsumption is a quasi-order whose equivalence is variable renaming, and strict-part is well-founded.

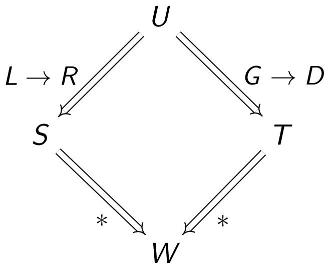
Subsumption extends naturally to context extensions. Given a set of unifiers, its minimal elements are said to be *most-general*.

Theorem

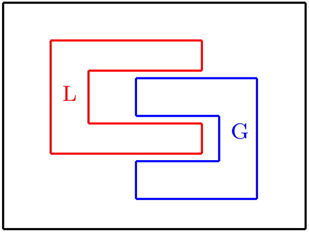
A solvable unification problem has a most general unifying extension computable in quadratic time and linear space.

Outline

Local Confluence



Overlap versus absence of overlap 1/2



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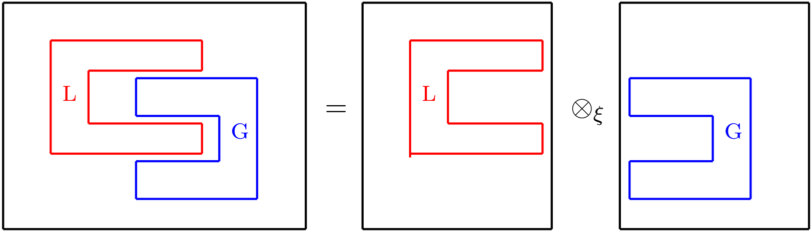

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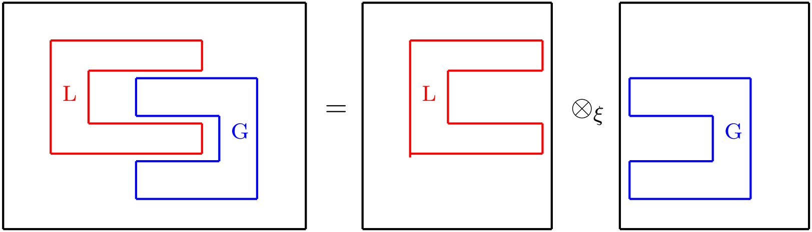
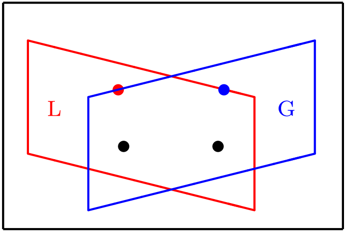
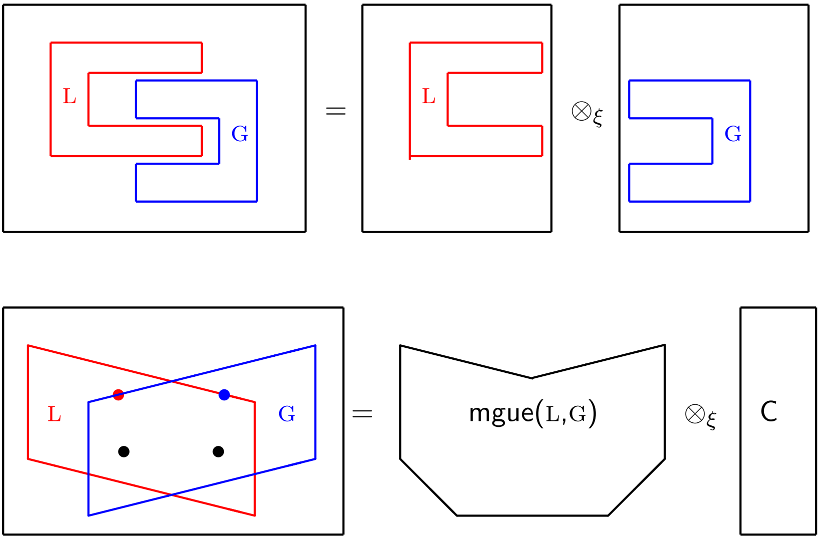




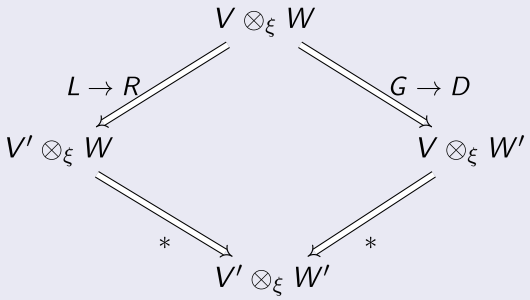
Overlap versus absence of overlap 1/2



Overlap versus absence of overlap 2/2

Lemma (Commutation)

Assume that $U = V \otimes_\xi W$, $V \underset{\mathrm{L \to R}}{\longrightarrow} V'$ *and* $W \underset{\mathrm{G \to D}}{\longrightarrow} W'$. *Then:*



Overlap versus absence of overlap 2/2

Lemma

Let $S \underset{L \to R}{\longleftarrow} U \underset{G \to D}{\longrightarrow} T$*, and let us assume that* U *has an internal vertex* w *which is an internal vertex of both* L *and* G*. Then, there exist* $\overline{u} \in \mathcal{V}er(\mathrm{L})$ *and* $\overline{v} \in \mathcal{V}er(\mathrm{G})$*, and a unifying extension* $\langle E, \zeta \rangle$ *of the equation* $\mathrm{L}[\overline{u}] = \mathrm{G}[\overline{v}]$ *such that* $U = \mathrm{L} \otimes_\zeta E = \mathrm{G} \otimes_\zeta E$*.*

Confluence reduces to critical pairs joinability

Definition (Critical pair)

Let $\mathrm{L} \rightarrow \mathrm{R}$ and $\mathrm{G} \rightarrow \mathrm{D}$ be two rules that are unifiable at partner vertices $\overline{v}, \overline{w}$, and $\langle C, \xi \rangle$ be an mgu. Then, $\langle \mathrm{L} \otimes_\xi C, \mathrm{G} \otimes_\xi C)$ is called a *critical pair* of $\mathrm{L}, \mathrm{G} \rightarrow \mathrm{D}$ at $\overline{v}, \overline{w}$.

Theorem

Let $\mathcal{S}$ be a terminating rewrite system on drags. Then, $\mathcal{S}$ is confluent iff all its critical pairs are joinable.

Achievements and Problems

- A rich algebra of drags based on COMPOSITION that provides a natural definition of rewriting

  Dershowitz&Jouannaud, *TCS*, 2019

- A rewrite ordering ordering over drags (wich is total up to permutation of roots)

  Dershowitz&Jouannaud, *LPAR*, 2018

- Drag unification is unitary (with quadratic complexity)

  Jouannaud&Orejas, *UNIF*, 2020

- Confluence of drag rewriting reduces to critical pairs joinability

  Jouannaud&Orejas, *IWC*, 2020

- Graph Rewriting versus Drag Rewriting

  Dershowitz, *Jouannaud&Orejas*, *Draft*, 2020

- Applications: logic programming, cyclic lambda-calculus, Koszul's completion of operads, etc.

Last slide

Thank you for your attention