Coherent Confluence in Modal n-Kleene Algebras

Cameron Calk

Laboratoire de l'École Polytechnique (LIX), Paris, France
cameron.calk@inria.fr

Abstract

This work concerns the algebraic formalisation of coherence and confluence. The Church-Rosser theorem states that the confluence of branchings and the confluence of zig-zags are equivalent properties. On the one hand, the coherent version of this theorem has been formulated in the language of higher dimensional globular categories. On the other hand, the Church-Rosser theorem has also been formulated using the structure of modal Kleene algebra. This work provides an algebraic formulation of the coherent Church-Rosser theorem. To this end, we introduce the structure of n-dimensional modal Kleene algebra satisfying globularity properties, providing a higher dimensional generalisation of modal Kleene algebra. In this setting, we give abstract definitions of coherent rewriting properties and, via modal operators, present novel proofs of the Church-Rosser theorem with higher-dimensional witnesses.

1 Introduction

In rewriting theory, a central theme is that of completing certain *branching shapes*, *i.e.* (local) branchings or zig-zags, with confluences, thus obtaining *confluence diagrams*. This is classically described with relations, but has more recently been formulated in the higher dimensional setting of polygraphs [2, 5]. While this approach is adapted to the study of string rewriting systems, it also provides a natural setting for the study of *coherence* [5] in abstract rewriting systems. Furthermore, a point-free algebraic formalisation of confluence is given in [7]. This abstracts the relational approach to the setting of Kleene algebras, providing a more general framework for the study of confluence. In this work, we combine these two approaches by introducing the structure of n-Kleene algebra, a natural setting for abstract, higher dimensional confluence proofs, and illustrate its use with the example of the Church-Rosser theorem.

In the relational setting, confluence is characterised, for an abstract rewriting system $\rightarrow$, *i.e.* a binary relation on some set, by the inclusion

$$\overset{*}{\leftarrow} \cdot \overset{*}{\rightarrow} \quad \subseteq \quad \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow},$$

where $\overset{*}{\rightarrow}$ denotes the reflexive, transitive closure of $\rightarrow$, $\leftarrow$ denotes its converse and $\cdot$ denotes relational composition. The Church-Rosser property is characterised by the inclusion

$$\overset{*}{\leftrightarrow} \quad \subseteq \quad \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow},$$

where $\overset{*}{\leftrightarrow} = (\leftarrow \cup \rightarrow)^*$ is the symmetric, reflexive, transitive closure of $\rightarrow$. The Church-Rosser theorem, which states that these two properties are equivalent, can be formulated with similar expressions in Kleene algebra using the *star* operation, an abstraction of the notion of reflexive, transitive closure, see Section 2.

The coherent formulation of this theorem takes place in the context of polygraphs. Roughly, it states that if there exists a set Γ of higher dimensional cells (with a globular shape) such that every branching can be completed to a confluence diagram filled by some element of Γ, then every zig-zag may be completed with a Church-Rosser diagram filled by a composition of elements in Γ, see [2]. In diagrams, these statements are respectively represented by:

where $\alpha \in \Gamma$ and β is a composition of elements of Γ. This is a coherence result akin to *e.g.* Mac Lane's theorem for monoidal categories, since local data, *i.e.* existence of confluence diagrams, is constructively used to prove a global result, *i.e.* existence of Church-Rosser diagrams.

In this abstract, we recall the structure of modal n-Kleene algebra from [2] and illustrate its use in the algebraic formalisation of coherence results by formulating and giving a proof sketch of the coherent Church-Rosser theorem. In Section 2 we provide background material, defining modal Kleene algebras as in [3] and Γ-coherence properties, an extension of constructions from [4]. Next, we introduce the novel structure of modal n-Kleene algebra and provide axioms for globularity in Section 3. Finally, in Section 4, we formalise the notion of coherence, state the Church-Rosser theorem in the setting of n-Kleene algebras and give a short proof sketch.

2 Modal Kleene Algebras and Coherence

Modal Kleene Algebras, [3]. Recall that a *dioid* is a semiring $(S, +, \cdot, 0, 1)$ in which addition is idempotent, *i.e.* for all $x \in S$, we have $x + x = x$. In this case, the natural partial order on S is defined by $x \leq y \;\Leftrightarrow\; x + y = y$. A *domain semiring* is a dioid $(S, +, \cdot, 0, 1)$ equipped with a *domain operation* $d : S \to S$ that satisfies, for all $x, y \in S$, the following axioms :

$$x \leq d(x)x \quad (1) \qquad d(xy) = d(xd(y)) \quad (2) \qquad d(x) \leq 1 \quad (3)$$

$$d(0) = 0 \quad (4) \qquad d(x + y) = d(x) + d(y) \quad (5)$$

where juxtaposition indicates multiplication. These axioms characterise domain in the case of relational algebras. For a more detailed account, see [3]. In particular, they imply that $S_d := d(S)$ forms a distributive lattice with the induced operations $+$ as join and $\cdot$ as meet, bounded by 0 and 1.

The *opposite* of a semiring S, in which the order of multiplication has been reversed, is denoted by S^{op}. A *codomain semiring* is a semiring equipped with a map $r : S \to S$ such that (S^{op}, r) is a domain semiring. A *modal semiring* is both a domain semiring and a codomain semiring, with additional axioms $d \circ r = r$ and $r \circ d = d$, which imply that $S_d = S_r$. We include these definitions for use in Section 3, in which the term *modal* will be justified.

A *(modal) Kleene algebra* is a (modal) semiring K equipped with an operator $(-)^* : K \to K$ called *Kleene star*. It satisfies, for all $x, y, z \in K$, the *unfold* and *induction* axioms

$$1 + xx^* \leq x^* \qquad 1 + x^*x \leq x^* \quad (6)$$

$$z + xy \leq y \Rightarrow x^*z \leq y \qquad z + yx \leq y \Rightarrow zx^* \leq y \quad (7)$$

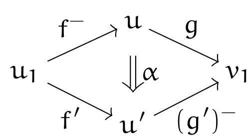
In a Kleene algebra K, the Church-Rosser theorem is expressed as follows: for elements $x, y \in K$, the following are equivalent:

$$x^*y^* \leq y^*x^* \qquad (x + y)^* \leq y^*x^* \quad (8)$$

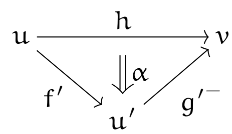
(x and y semi-commute) (x and y are Church-Rosser.)

Γ-Coherence Properties. Recall that an n-polygraph P is a data consisting of sets $(P_k)_{0 \leq k \leq n}$ and *source* and *target* maps $(s_k, t_k : P_{k+1} \to P_k^*)_{0 \leq k < n}$ satisfying globularity conditions, where P_k^* is the free k-category generated by the globular set $(P_0, P_1^*, \dots, P_{k-1}^*, P_k)$. A *cellular extension* of P is a set Γ and (globular) attaching maps $s_n, t_n : \Gamma \to P_n^\top$, where $P_n^\top$ is the free $(n, n-1)$-category generated by P. See [5] for a more detailed account of polygraphs and their use in rewriting theory.

Given an n-polygraph P and a cellular extension Γ, we say that P is Γ-confluent (*resp.* Γ-Church-Rosser) if, for every branching $(f, g) \in (P_n^*)^2$ of n-cells (*resp.* every zig-zag $h \in P_n^\top$), there exist a confluence (f', g') and a $(n+1)$-cell $\alpha \in \Gamma$ (*resp.* $\alpha \in P_n^\top(\Gamma)$, the free $(n+1, n-1)$-category generated by (P, Γ)) such that



$\alpha : f^- \star_{n-1} g \to f' \star_{n-1} (g')^-$



(*resp.* $\alpha : f \to f' \star_{n-1} g'^-$),

where, for n-cells f_1, f_2 of P, $f_1 \star_{n-1} f_2$ denotes their composition with respect to the $(n-1)$-dimensional target (*resp.* source) of f_1 (*resp.* f_2). Readers familiar with polygraphic rewriting may notice that the filling cells are usually oriented from left to right, rather than from top to bottom. We choose this orientation to better reflect the approach used in the setting of Kleene algebras, in which branching shapes, such as x^*y^* and $(x+y)^*$, are related to confluence shapes y^*x^*, as recalled in (8). This will equally be the case in n-Kleene algebra, see Section 4.

The coherent Church-Rosser theorem [2] is formulated as follows: given P a n-polygraph, and Γ a cellular extension of P, if P is Γ-confluent, then P is Γ-Church-Rosser.

3 Higher Dimensional Modal Kleene Algebra

n-Dioids. For $n \geq 1$, an n-*dioid* is a structure $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$ such that $(S, +, 0, \odot_i, 1_i)$ is a dioid for $0 \leq i < n$, the *lax interchange change laws* hold, and units are idempotent with respect to lower dimensional multiplications, *i.e.*

$$(A \odot_j A') \odot_i (B \odot_j B') \leq (A \odot_i B) \odot_j (A' \odot_i B') \quad \text{and} \quad 1_j \odot_i 1_j = 1_j, \tag{9}$$

for all $A, A', B, B' \in S$ and $0 \leq i < j < n$. The opposite n-semiring of S, denoted S^{op}, is that in which the order of every multiplication operation has been reversed.

Globular n-Semirings with Domains. A *domain n-semiring* is an n-dioid $(S, +, 0, \odot_i, 1_i)_{0 \leq i < n}$ equipped with n domain maps $(d_i : S \to S)_{0 \leq i < n}$, such that $(S, +, 0, \odot_i, 1_i, d_i)$ is a domain semiring and $d_{i+1} \circ d_i = d_i$ for all $0 \leq i < n-1$. A n-semiring with codomains is equipped with maps $(r_i : S \to S)_{0 \leq i < n}$ such that S^{op} is a domain n-semiring with respect to the $(r_i)_{0 \leq i < n}$.

A *modal n-semiring* is an n-semiring with domains and codomains, in which the coherence conditions $d_i \circ r_i = r_i$ and $r_i \circ d_i = d_i$ hold for all $0 \leq i < n$. Given S a modal n-semiring, we define *forward* and *backward i-diamond* operators defined via (co-)domain operators in each dimension, as in [3]. For any $0 \leq i < n$, $A \in S$ and $\phi \in S_i$, we define

$$|A\rangle_i(\phi) = d_i(A \odot_i \phi), \quad \text{and} \quad \langle A|_i(\phi) = r_i(\phi \odot_i A). \tag{10}$$

These are modal operators on the distributive lattice S_i in the sense of [6]. A modal semiring S is called *globular* if the following *globular relations* hold for $0 \leq i < j < n$ and $A, B \in K$:

$$d_i \circ d_j = d_i \text{ and } d_i \circ r_j = d_i, \tag{11}$$

$$r_i \circ d_j = r_i, \text{ and } r_i \circ r_j = r_i, \tag{12}$$

$$d_j(A \odot_i B) = d_j(A) \odot_i d_j(B), \tag{13}$$

$$r_j(A \odot_i B) = r_j(A) \odot_i r_j(B). \tag{14}$$

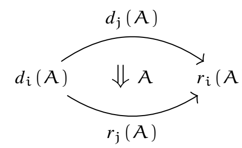
Modal n-Kleene Algebra. A n-*Kleene algebra* is a n-semiring K equipped with maps $(-)^{*_i} : K \to K$

such that $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a Kleene algebra for $0 \le i < n$, and for $0 \le i < j < n$, the *Kleene plus* operator $(-)^{+_j}$, defined by $A^{+_j} = A \odot_j A^{*_j}$, is a lax morphism with respect to the i-multiplication of j-dimensional elements on the right (resp. left) in the sense that for all $A \in K$ and $\phi \in K_j$,

$$\phi \odot_i A^{+_j} \le (\phi \odot_i A)^{+_j}, \qquad \text{and} \qquad (\text{resp. } A^{+_j} \odot_i \phi \le (A \odot_i \phi)^{+_j}). \tag{15}$$

When the underlying semiring is globular and modal, we say that K is a *globular modal n-Kleene algebra*.

To provide a link to the polygraphic case, we remark that the power set of the set of n-cells in the free n-category generated by an n-polygraph P constitutes a globular modal n-Kleene algebra $K(P)$. Following this intuition, an element A of a globular modal n-Kleene algebra will be represented with respect to its i- and j-(co)domains as in the adjacent diagram.



4 Church-Rosser theorem

Let K be a globular modal n-Kleene algebra and $0 \le i < j < n$. Given elements ϕ and ψ of K_j, we say that $A \in K$ is a i-*confluence filler* for (ϕ, ψ) if $|A\rangle_j(\psi^{*_i} \odot_i \phi^{*_i}) \ge \phi^{*_i} \odot_i \psi^{*_i}$.

In the modal n-Kleene algebra $K(P)$ corresponding to a polygraph P, taking $i = n-2$, $j = n-1$, letting ψ be the set of $(n-1)$-cells of P and letting ϕ be the set of their inverses, this signifies that A is a set of n-cells containing a filling cell for a confluence diagram corresponding to each branching of $(n-1)$-cells.

4.1. Theorem (Coherent Church-Rosser in globular n-MKA). *Let K be a globular modal n-Kleene algebra. Given $\phi, \psi \in K_j$, for $0 < j < n$, for any i-confluence filler $A \in K$ for (ϕ, ψ), we have*

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \ge (\phi + \psi)^{*_i},$$

where $\hat{A} := (\phi + \psi)^{*_i} \odot_i A \odot_i (\phi + \psi)^{*_i}$.

Note that the expression $|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \ge (\phi + \psi)^{*_i}$ signifies that the domain of $\hat{A}^{*_j}$, when restricted on the right to confluences $\psi^{*_i}\phi^{*_i}$, contains at least all of the zig-zags $(\phi + \psi)^{*_i}$.

To finish this section, we provide a sketch of the proof, a detailed version of which can be found in [2]. To ease notation, juxtaposition will denote i-multiplication. By (7) and (15), we have

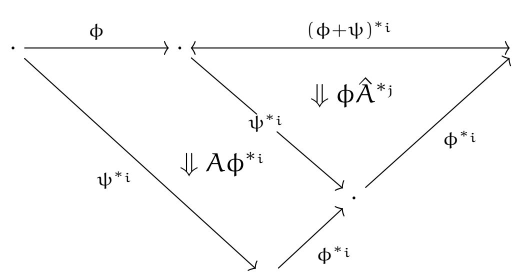
$$1_i + (\phi + \psi)|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \le |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \;\Rightarrow\; (\phi + \psi)^{*_i} \le |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$$

Observing that $1_i \le \psi^{*_i}\phi^{*_i} \le |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})$, it remains to show that

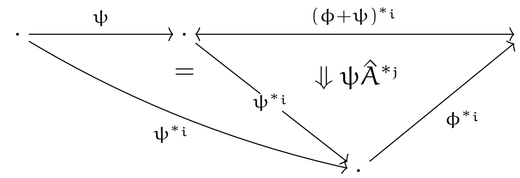
$$(\phi + \psi)|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) \le |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}).$$

By distributivity, we may prove this for each of the summands. Below, we show the formal calculations in the n-Kleene algebra on the left, which are reflected in the diagrammatic representations on the right:

$$\begin{aligned}
\phi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) &\le |\phi\hat{A}^{*_j}\rangle_j(\phi\psi^{*_i}\phi^{*_i}) \\
&\le |\phi\hat{A}^{*_j}\rangle_j(|A\rangle_j(\psi^{*_i}\phi^{*_i})\phi^{*_i}) \\
&\le |\phi\hat{A}^{*_j}\rangle_j(|A\phi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}\phi^{*_i})) \\
&\le |\phi\hat{A}^{*_j} \odot_j A\phi^{*_i}\rangle_j(\psi^{*_i}\phi^{*_i}) \\
&\le |\hat{A}^{*_j} \odot_j \hat{A}\rangle_j(\psi^{*_i}\phi^{*_i}) \\
&\le |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})
\end{aligned}$$



$$\begin{aligned}\psi|\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i}) &\leq |\psi\hat{A}^{*_j}\rangle_j(\psi\psi^{*_i}\phi^{*_i})\\ &\leq |\psi\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})\\ &\leq |\hat{A}^{*_j}\rangle_j(\psi^{*_i}\phi^{*_i})\end{aligned}$$



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5 Conclusion

By introducing the structure of modal n-Kleene algebra, we have provided an algebraic framework for coherent confluence proofs in higher dimensional rewriting theory. A more detailed account of this structure and its properties can be found in [2]. There, in addition to the Church-Rosser theorem, a notion of termination in modal n-Kleene algebras is formalised, and a coherent version of Newman's lemma is formulated and proved in the setting of globular modal n-Kleene algebra.

Perspectives for future work most notably include a formulation of the coherent confluence Squier theorem [5] for abstract rewriting systems in globular modal n-Kleene algebras. This result concerns homotopical properties of polygraphs considered as cofibrant objects in the folk model structure of ω-categories. A central goal is thus to formalise the notion of cofibrant object in the language of modal n-Kleene algebras, and thereby relate these to questions in homotopy type theory. Other avenues of future work include formalising the structure of modal n-Kleene algebra in the proof assistant Isabelle, following [1], and developing a cubical approach.

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