Parallel Closedness Revisited

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Abstract

In this note we present a simple proof of Huet's parallel closedness theorem (1980). We also show that Toyama's almost parallel closedness (1988), a generalization of Huet's theorem, is subsumed by his earlier result (1981), which is known to be a generalization of Gramlich's confluence criterion based on parallel critical pairs (1996).

1 Introduction

The parallel closedness theorem by Huet [3] is a pioneering work in analyzing confluence of left-linear term rewrite systems (TRSs). It claims that a left-linear TRS is confluent if every critical pair is closed by parallel step $\twoheadrightarrow\!\!\!\!\!\!\!\;\shortmid\;\;$ ($\mathrel{+\!\!\!\!\!\!\rightarrow}$). His theorem, including its ingenious proof technique, had a significant influence on later confluence research. In 1996, Gramlich [2] introduced *parallel* critical pairs to make a powerful variant of Huet's theorem. While the resulting criterion enables us to use relaxed forms for the closing condition, he also showed that it is not comparable with Huet's theorem. This might give us the impression that criteria based on parallel critical pairs cannot supersede ones based on ordinary critical pairs (see e.g. [1, Section 6.5]). However, this is wrong. We will show that Toyama's earlier result [9] subsumes Huet's theorem as well as Gramlich's criterion. Actually it subsumes the almost parallel closedness theorem [10], which is Toyama's later work and known as a generalization of Huet's theorem.

In the remaining part of the note we revisit Huet's and Toyama's parallel closedness theorems. Introducing a new induction measure, we give a simple proof of the parallel closedness theorem in Section 2. In Section 3 we show that Toyama's earlier result subsumes his later result. We assume familiarity with term rewriting [1, 8].

2 Huet's Parallel Closedness

In this section we present a simple proof of Huet's parallel closedness theorem [3] with a new measure. By denoting the critical pair $t \leftarrow \cdot \xrightarrow{\epsilon} u$ by $t \leftarrow\!\rtimes\!\xrightarrow{\epsilon} u$, the theorem is stated as follows:

Definition 1 ([3])**.** *A TRS is* parallel closed *if* $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!\!\rightarrow}$.

Theorem 1 ([3])**.** *A left-linear TRS is confluent if it is parallel closed.*

We prepare notations for our new measure. Let t be a term. The size of t is denoted by $|t|$. Given a set P of positions in t, we write $|t|_P$ for the sum of $|(t|_p)|$ for all $p \in P$. Note that $|t| \geqslant |t|_P$ holds if P is a set of parallel positions of t. We define the strict order $\succ$ as the lexicographic product of the standard order $>$ on $\mathbb{N}$ and the proper superterm relation $\rhd$. We also prepare terminologies for analyzing peaks. Let $\Gamma : t \overset{P}{\leftarrow\!\!\!\!+\!\!\!\!+} \ell\sigma \xrightarrow{\epsilon} r\sigma$ be a peak, where $\ell \to r$ is a rewrite rule. We say that Γ is *overlapping* if some position $p \in P$ is a function position in ℓ and in the case of $P = \{\epsilon\}$ the rule employed in $t \overset{P}{\leftarrow\!\!\!\!+\!\!\!\!+} \ell\sigma$ is not a variant of $\ell \to r$. Otherwise, Γ is *non-overlapping*. We are ready for proving the theorem.

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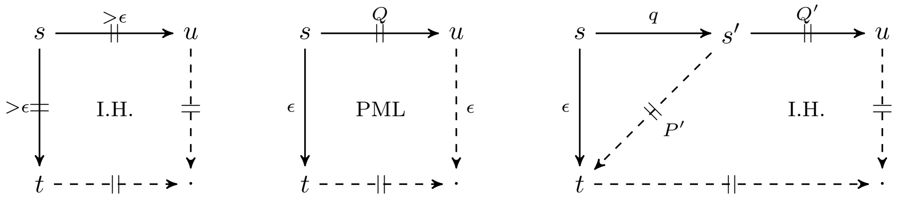


(ii) non-root (iii) non-overlapping (iv) overlapping

Figure 1: Proof of Theorem 1.

Proof of Theorem 1. Let $\mathcal{R}$ be a left-linear and parallel closed TRS. It is sufficient to show that $\mathrel{\shortparallel\!\!\!\to}$ has the diamond property. Let $\Gamma\colon t \overset{P}{\mathrel{\leftarrow\!\!\!\shortparallel}} s \overset{Q}{\mathrel{\shortparallel\!\!\!\to}} u$ be a peak. By well-founded induction on $(|t|_P + |u|_Q, s)$ with respect to $\succ$ we show $t \mathrel{\shortparallel\!\!\!\to} \cdot \mathrel{\leftarrow\!\!\!\shortparallel} u$. Depending on the shape of Γ, we distinguish four cases. Figure 1 illustrates cases (ii)–(iv).

(i) If P or Q is empty then $t \mathrel{\shortparallel\!\!\!\to} u$ or $t \mathrel{\leftarrow\!\!\!\shortparallel} u$. In either case the claim holds.

(ii) If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then Γ is of form $f(t_1,\dots,t_n) \mathrel{\leftarrow\!\!\!\shortparallel} f(s_1,\dots,s_n) \mathrel{\shortparallel\!\!\!\to} f(u_1,\dots,u_n)$ with $t_i \overset{P_i}{\mathrel{\leftarrow\!\!\!\shortparallel}} s_i \overset{Q_i}{\mathrel{\shortparallel\!\!\!\to}} u_i$ for all $1 \leqslant i \leqslant n$. Here $P_i = \{p \mid ip \in P\}$ and $Q_i = \{q \mid iq \in Q\}$. For each $i \in \{1,\dots,n\}$, we have $|t|_P \geqslant |t_i|_{P_i}$ and $|u|_Q \geqslant |u_i|_{Q_i}$, and therefore $|t|_P + |u|_Q \geqslant |t_i|_{P_i} + |u_i|_{Q_i}$. Since $(|t|_P + |u|_Q, s) \succ (|t_i|_{P_i} + |u_i|_{Q_i}, s_i)$ holds, the induction hypothesis yields $t_i \mathrel{\shortparallel\!\!\!\to} v_i \mathrel{\leftarrow\!\!\!\shortparallel} u_i$ for some v_i. Thus, $t \mathrel{\shortparallel\!\!\!\to} f(v_1,\dots,v_n) \mathrel{\leftarrow\!\!\!\shortparallel} u$ follows.

(iii) If Γ is non-overlapping and P or Q is $\{\epsilon\}$ then the Parallel Moves Lemma applies [1, Lemma 6.4.4]. Figure 1(iii) illustrates the case of $P = \{\epsilon\}$.

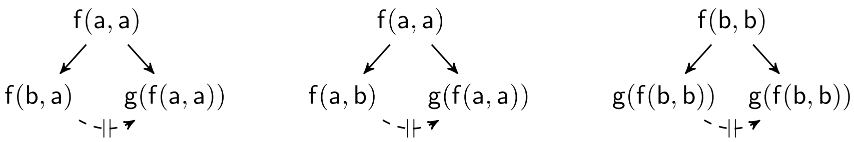
(iv) If Γ is overlapping and P or Q is $\{\epsilon\}$, say $P = \{\epsilon\}$, then there exists an overlapping peak $t \overset{\epsilon}{\leftarrow} s \overset{q}{\rightarrow} s'$ with $s' \overset{Q'}{\mathrel{\shortparallel\!\!\!\to}} u$ for $Q' = Q \setminus \{q\}$. By parallel closedness we have $s' \overset{P'}{\mathrel{\shortparallel\!\!\!\to}} t$ for some P'. As $|t|_{\{\epsilon\}} \geqslant |t|_{P'}$ and $|u|_Q > |u|_{Q'}$ hold, $|t|_{\{\epsilon\}} + |u|_Q > |t|_{P'} + |u|_{Q'}$ holds. Since $(|t|_{\{\epsilon\}} + |u|_Q, s) \succ (|t|_{P'} + |u|_{Q'}, s')$ holds, the induction hypothesis yields $t \mathrel{\shortparallel\!\!\!\to} \cdot \mathrel{\leftarrow\!\!\!\shortparallel} u$. $\square$

With a small example taken from [2], we illustrate the usage of the theorem.

Example 1. *Consider the left-linear and non-terminating TRS [2]:*

$$\mathsf{a} \to \mathsf{b} \qquad \mathsf{f}(\mathsf{a},\mathsf{a}) \to \mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a})) \qquad \mathsf{f}(\mathsf{b},x) \to \mathsf{g}(\mathsf{f}(x,x)) \qquad \mathsf{f}(x,\mathsf{b}) \to \mathsf{g}(\mathsf{f}(x,x))$$

While the TRS admits three critical peaks, all of them are closed by single parallel steps:



Thus, the TRS is parallel closed. Hence, the TRS is confluent.

Using the TRS in Example 1, we compare our induction measure with Huet's original measure [3]. Our proof measures a peak by the amount of *contractums*. Consider e.g. the peak:

$$\mathsf{f}(\overline{\mathsf{b}},\overline{\mathsf{b}}) \overset{\{1,2\}}{\mathrel{\leftarrow\!\!\!\shortparallel}} \underline{\mathsf{f}(\underline{\mathsf{a}},\underline{\mathsf{a}})} \overset{\{\epsilon\}}{\mathrel{\shortparallel\!\!\!\to}} \overline{\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a}))}$$

The overlined part indicates the contractums in the target terms of the parallel steps.[1] So the amount is $|\mathsf{b}| + |\mathsf{b}| + |\mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a}))| = 6$. Huet's proof [3, 1] measures a peak $t \overset{P_1}{\mathrel{\leftarrow\!\!\!\!\!\parallel}} s \overset{P_2}{\mathrel{\parallel\!\!\!\!\!\rightarrow}} u$ by the amount of overlaps of *redexes*:

$$|s, P_1, P_2| := \sum_{p \in Q_1 \cup Q_2} \big|(s|_p)\big|$$

where $Q_1 = \{p \in P_1 \mid p \geqslant q \text{ for some } q \in P_2\}$ and $Q_2 = \{p \in P_2 \mid p > q \text{ for some } q \in P_1\}$. In the above peak the doubly underlined part indicates the overlaps of the redexes. So the amount is 2. Even with this measure the same proof goes through, provided that well-founded induction on $(|s,P,Q|, s)$ with respect to $\succ$ is performed. However, proving the inequality $|s,\{\epsilon\},Q| > |s',P',Q'|$ in case (iv) is notoriously difficult [5, 4]. This part is trivial in our proof.

3 Toyama's Extensions

Toyama made two variations of Huet's parallel closedness theorem in 1981 [9] and in 1988 [10], but their relation has not been known. In this section we briefly recall his and related results, and then show that Toyama's earlier result subsumes the later one.

In 1988, Toyama showed that the closing form for *overlay* critical pairs, originating from root overlaps, can be relaxed. Let $t \overset{p}{\leftarrow} \cdot \overset{\epsilon}{\rightarrow} u$ be a critical pair. We write $t \overset{\epsilon}{\leftarrow}\!\rtimes\!\rightarrow u$ if $p = \epsilon$, and $t \overset{>\epsilon}{\longleftarrow}\!\rtimes\!\rightarrow u$ if $p > \epsilon$.

Definition 2 ([10]). *A TRS is* almost parallel closed *if the inclusions* $\overset{\epsilon}{\leftarrow}\!\rtimes\!\rightarrow \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow$ *and* $\overset{>\epsilon}{\longleftarrow}\!\rtimes\!\rightarrow \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow}$ *hold.*

Theorem 2 ([10]). *A left-linear TRS is confluent if it is almost parallel closed.*

Inspired by almost parallel closedness, Gramlich [2] developed a confluence criterion based on *parallel critical pairs* in 1996.

Definition 3. *We say that* $(\ell\sigma)[r_p]_{p \in P} \mathrel{\leftarrow\!\!\!\!\!\parallel} \ell\sigma \overset{\epsilon}{\rightarrow} r\sigma$ *is a* parallel critical peak *of a TRS* $\mathcal{R}$ *if*

- $P \subseteq \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$ *is a non-empty set of parallel positions in* ℓ,
- $\ell \to r$ *and* $\ell_p \to r_p$ *(for* $p \in P$*) are variants of* $\mathcal{R}$*-rules having no common variables,*
- σ *is a most general unifier of* $\{\ell_p \approx (\ell|_p)\}_{p \in P}$*, and*
- *if* $P = \{\epsilon\}$ *then* $\ell_\epsilon \to r_\epsilon$ *is not a variant of* $\ell \to r$.

We write $t \overset{>\epsilon}{\mathrel{\leftarrow\!\!\!\!\!\parallel}}\!\rtimes\!\rightarrow u$ *if* $t \overset{P}{\mathrel{\leftarrow\!\!\!\!\!\parallel}} s \overset{\epsilon}{\rightarrow} u$ *is a parallel critical peak and* $P \neq \{\epsilon\}$.

Theorem 3 ([2]). *A left-linear TRS is confluent if the inclusions* $\leftarrow\!\rtimes\!\overset{\epsilon}{\rightarrow} \subseteq \mathrel{\parallel\!\!\!\!\!\rightarrow} \cdot {}^{*}\!\!\leftarrow$ *and* $\overset{>\epsilon}{\mathrel{\leftarrow\!\!\!\!\!\parallel}}\!\rtimes\!\rightarrow \subseteq \rightarrow^{*}$ *hold.*

Unfortunately, this criterion by Gramlich does not subsume (almost) parallel closedness.

Example 2 (Continued from Example 1). *The TRS admits the following non-overlay parallel critical peak* $\mathsf{f}(\mathsf{b},\mathsf{b}) \mathrel{\leftarrow\!\!\!\!\!\parallel} \mathsf{f}(\mathsf{a},\mathsf{a}) \overset{\epsilon}{\rightarrow} \mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a}))$*. However,* $\mathsf{f}(\mathsf{b},\mathsf{b}) \rightarrow^{*} \mathsf{g}(\mathsf{f}(\mathsf{a},\mathsf{a}))$ *does not hold.*

[1] Up to our best knowledge, this idea first appeared in the paper by Oyamaguchi and Ohta [7].

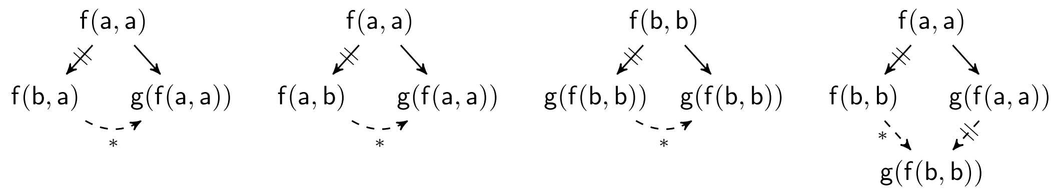
As noted in the paper [2], Toyama [9] had already obtained in 1981 a closedness result that subsumes Theorem 3.

Theorem 4 ([9])**.** *A left-linear TRS is confluent if the following conditions hold.*

(a) The inclusion $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{\shortparallel\!\!\!\!\!\rightarrow} \cdot {}^*\!\leftarrow$ *holds.*

(b) Every parallel critical peak $t \overset{P}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s \xrightarrow{\epsilon} u$ *admits a valley of the form* $t \rightarrow^* v \overset{Q}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} u$ *with* $\mathcal{V}\mathsf{ar}(s, P) \supseteq \mathcal{V}\mathsf{ar}(v, Q)$*. Here* $\mathcal{V}\mathsf{ar}(s, P)$ *stands for* $\bigcup_{p \in P} \mathcal{V}\mathsf{ar}(s|_p)$*.*

Example 3. *Consider again the TRS of Example 1. As witnessed in Example 1, the inclusion* $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{\shortparallel\!\!\!\!\!\rightarrow}$ *holds. Thus, condition (a) of Theorem 4 holds. There are four parallel critical peaks and they can be closed as follows:*



It is easy to see that the diagrams fulfil condition (b). Hence, confluence follows by Theorem 4.

We show that Theorem 4 even subsumes Theorem 2. The next lemma relates the Parallel Moves Lemma [1, Lemma 6.4.4] to the variable condition of Theorem 4.

Lemma 1. *Let* $\mathcal{R}$ *be a left-linear TRS. If* $t \overset{P}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s \xrightarrow{\epsilon} u$ *is non-overlapping then* $t \xrightarrow{\epsilon} v \overset{Q}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} u$ *and* $\mathcal{V}\mathsf{ar}(s, P) \supseteq \mathcal{V}\mathsf{ar}(v, Q)$ *for some term* v *and set* Q *of parallel positions.*

The above statement is extended to parallel peaks for almost parallel closed TRSs.

Lemma 2. *Let* $\mathcal{R}$ *be a left-linear almost parallel closed TRS. If* $t \overset{P_1}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s \overset{P_2}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} u$ *then*

- $t \rightarrow^* v_1 \overset{Q_1}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} u$ *and* $\mathcal{V}\mathsf{ar}(s, P_1) \supseteq \mathcal{V}\mathsf{ar}(v_1, Q_1)$ *for some* v_1 *and* Q_1*, and*
- $t \overset{Q_2}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} v_2 \ {}^*\!\leftarrow u$ *and* $\mathcal{V}\mathsf{ar}(s, P_2) \supseteq \mathcal{V}\mathsf{ar}(v_2, Q_2)$ *for some* v_2 *and* Q_2*.*

Proof. Let $\Gamma\colon t \overset{P_1}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s \overset{P_2}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} u$. We perform well-founded induction on $(|t|_{P_1} + |u|_{P_2}, s)$ with respect to $\succ$. We distinguish cases, depending on the shape of Γ.

(i) If P_1 or P_2 is $\varnothing$ then the claim follows from the fact: $\mathcal{V}\mathsf{ar}(w, P) \supseteq \mathcal{V}\mathsf{ar}(v, P)$ if $w \overset{P}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} v$.

(ii) If $P_1 \nsubseteq \{\epsilon\}$ and $P_2 \nsubseteq \{\epsilon\}$ then we may assume $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $u = f(u_1, \ldots, u_n)$, and $t_i \overset{P_1^i}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s_i \overset{P_2^i}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} u_i$ for all $1 \leqslant i \leqslant n$. Here P_k^i denotes the set $\{p \mid i \cdot p \in P_k\}$. As in case (ii) of the proof of Theorem 1 we can deduce $(|t|_{P_1} + |u|_{P_2}, s) \succ (|t_i|_{P_1^i} + |u_i|_{P_2^i}, s_i)$. Consider an i-th peak $t_i \overset{P_1^i}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} s_i \overset{P_2^i}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} u_i$. By the induction hypothesis it admits valleys of the forms $t_i \overset{Q_1^i}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} v_1^i \ {}^*\!\leftarrow u_i$ and $t_i \rightarrow^* v_2^i \overset{Q_2^i}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} u_i$ such that $\mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \mathcal{V}\mathsf{ar}(s_i, P_k^i)$ for both $k \in \{1, 2\}$. For each k, take $Q_k = \{i \cdot q \mid 1 \leqslant i \leqslant n \text{ and } q \in Q_k^i\}$ and $v_k = f(v_k^1, \ldots, v_k^n)$. Then, $t \rightarrow^* v_1 \overset{Q_1}{\mathrel{\leftarrow\!\!\!\!\!\shortparallel}} u$ and $t \overset{Q_2}{\mathrel{\shortparallel\!\!\!\!\!\rightarrow}} v_2 \ {}^*\!\leftarrow u$ hold, and moreover the inclusion

$$\mathcal{V}\mathsf{ar}(v_k, Q_k) = \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(s_i, P_k^i) = \mathcal{V}\mathsf{ar}(s, P_k)$$

holds for each k. Hence, the claim follows.

(iii) If Γ is non-overlapping and P_1 or P_2 is $\{\epsilon\}$ then the claim is straightforward from Lemma 1.

(iv) If Γ is overlapping with $P_1 = P_2 = \{\epsilon\}$ then by almost parallel closedness $t \to^* v_1 \overset{Q_1}{\leftarrow\!\!\!\!\!\!\parallel} u$ and $t \overset{Q_2}{\parallel\!\!\!\!\!\!\rightarrow} v_2 \;{}^*\!\!\leftarrow u$ for some v_1, v_2, Q_1, and Q_2. For each $k \in \{1, 2\}$ we have $s \to^* v_k$, so $\mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s)$ follows. Therefore, $\mathcal{V}\mathsf{ar}(v_k, Q_k) \subseteq \mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s) = \mathcal{V}\mathsf{ar}(s, \{\epsilon\})$. The claim holds.

(v) If Γ is overlapping with $P_1 = \{\epsilon\}$ and $P_2 \neq \{\epsilon\}$ then there exists $p \in P_2$ such that $s \xrightarrow{p} s' \overset{P_2\setminus\{p\}}{\parallel\!\!\!\!\!\!\rightarrow} u$ and $t \xleftarrow{\epsilon} s \xrightarrow{p} s'$ is an instance of a critical peak. By almost parallel closedness $t \overset{P_1'}{\leftarrow\!\!\!\!\!\!\parallel} s'$ for some P_1'. As in case (iv) of the proof of Theorem 1 we can deduce $(|t|_{P_1} + |u|_{P_2}, s) \succ (|t|_{P_1'} + |u|_{P_2\setminus\{\epsilon\}}, s')$. Thus, the claim follows by the induction hypothesis for $t \overset{P_1'}{\leftarrow\!\!\!\!\!\!\parallel} s' \overset{P_2\setminus\{p\}}{\parallel\!\!\!\!\!\!\rightarrow} u$ and $\mathcal{V}\mathsf{ar}(s', P_2 \setminus \{p\}) \subseteq \mathcal{V}\mathsf{ar}(s, P_2)$. □

Theorem 5. *Every left-linear and almost parallel closed TRS satisfies conditions (a) and (b) of Theorem 4. In other words, Theorem 4 subsumes Theorem 2.*

Proof. Since (parallel) critical peaks are instances of $\leftarrow\!\!\!\!\!\!\parallel \cdot \parallel\!\!\!\!\!\!\rightarrow$, Lemma 2 entails the claim. □

4 Concluding Remark

We presented a simple proof of Huet's parallel closedness theorem, using a new measure, and also proved that Toyama's almost parallel closedness theorem is subsumed by his earlier result based on parallel critical pairs. We anticipate that our measure can be adapted to other confluence criteria for term rewriting and conditional rewriting, while it is unclear whether it can be adapted to the proof of van Oostrom's development closedness [6]. At least the current definition does not go through. Despite its powerfulness, Theorem 4 has not been well studied. In particular, whether the variable condition of Theorem 4 is essential is our primary question.

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